Notes on Convolution

October 24, 2024

1 Notation

In discrete time, consider two signals $f[i]$, $g[i]$. Their convolution $*$ is defined as

$$(f * g)[n] = \sum_{m=-M}^{m=M} f[n-m]g[m]$$

with support $\{-M, -M+1, \ldots, M-1, M\}$. Outside of the support, the signal is zero.
In continuous time, consider two functions $f(x)$, $g(x)$. Their convolution is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(u)g(x-u)\mathrm{d}u$$

2 Properties

Commutation

$$f * g = g * f \tag{1}$$

Proof (through a change of variables):

$$(f * g)(x) = \int_{-\infty}^{\infty} f(u)g(x-u)\mathrm{d}u = \int_{-\infty}^{\infty} f(x-v)g(v)\mathrm{d}v = \int_{-\infty}^{\infty} g(v)f(x-v)\mathrm{d}v = (g * f)(x)$$

Dirac Delta
Let $\delta(x)$ denote the dirac delta function, defined as

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \text{undefined} & x = 0 \end{cases}, \qquad \int_{-\infty}^{\infty} \delta(u)\mathrm{d}u = 1$$

$$f * \delta = f \tag{2}$$

Proof (through using the definition):

$$(f * \delta)(x) = \int_{-\infty}^{\infty} f(u)\delta(x-u)\mathrm{d}u = f(x)$$

Derivatives

$$(f * g)' = f' * g \overset{(1)}{=} g' * f \tag{3}$$

Proof:

$$(f * g)' = \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{\infty} f(u)g(x-u)\mathrm{d}u = \int_{-\infty}^{\infty} f(u)\frac{\mathrm{d}}{\mathrm{d}x}g(x-u)\mathrm{d}u = f * g'$$