SVD for DLT

Why does it work?

Leonard Bauersfeld

October 5, 2023

1 Proof

Nullspace

In a DLT problem with N points, we need to find the solution the following system of equations (and ignore the trivial solution $\mathbf{m} = 0$):

$$\mathbf{Qm} = \mathbf{0}, \quad \text{where } \mathbf{Q} \in \mathbb{R}^{2N \times 12},\ \mathbf{m} \in \mathbb{R}^{12},\ \mathbf{m} \neq \mathbf{0} \tag{1}$$

Mathematically, this means that the vector $\mathbf{m}$ has to lie in the *nullspace* of $\mathbf{Q}$:

$$\mathbf{Qm} = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{m} \in \text{Null}(\mathbf{Q})$$

Note that, for any $\mathbf{m} \in \text{Null}(\mathbf{Q})$, any α-multiple also lies in the nullspace, i.e.

$$\mathbf{Qm} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{Q}\alpha\mathbf{m} = 0, \quad \alpha \in \mathbb{R} \tag{2}$$

Minimization Problem

Using the fact that (any) norm of a vector is greater or equal to zero, we can reformulate eq. (1) as a minimization problem as follows:

$$\begin{aligned} &\min_{\mathbf{m}} ||\mathbf{Qm}|| \\ &\text{s.t. } ||\mathbf{m}|| = 1 \end{aligned} \tag{3}$$

The constraint on the norm of $||\mathbf{m}||$ ensures that the trivial solution is ignored. Note that it does not change the solution, because of the scale-invariance of the solution, see eq. (2).

If a solution to eq. (1) exists, the minimization above will find it. If no (exact) solution exists, the minimization problem will find the *best* possible solution, in the sense of the norm. If we chose the norm $||\cdot||$ to be the L2-norm, we obtain a solution that is optimal in the least-squares sense.

Solution to the minimization

To solve the minimization problem eq. (3), we use the properties of the singular value decomposition to simplify the problem and then solve it *by inspection.*

$$\begin{aligned} &\min_{\mathbf{m}} ||\mathbf{Qm}|| && \text{let } \mathbf{Q} = \mathbf{USV}^\top \\ &\text{s.t. } ||\mathbf{m}|| = 1 && \text{with } \mathbf{U} \in \mathbb{R}^{2N\times 12},\ \mathbf{S} \in \mathbb{R}^{12\times 12},\ \mathbf{V} \in \mathbb{R}^{12\times 12} \end{aligned} \tag{4}$$

$$\begin{aligned} &\min_{\mathbf{m}} ||\mathbf{USV}^\top\mathbf{m}|| && \mathbf{U} \text{ is orthonormal (no change to length of vector)} \\ &\text{s.t. } ||\mathbf{m}|| = 1 && \end{aligned} \tag{5}$$

$$\begin{aligned} &\min_{\mathbf{m}} ||\mathbf{SV}^\top\mathbf{m}|| && \text{let } \mathbf{y} = \mathbf{V}^\top\mathbf{m} \\ &\text{s.t. } ||\mathbf{m}|| = 1 && \mathbf{Vy} = \mathbf{VV}^\top\mathbf{m} = \mathbf{m} \text{ because V is orthnormal} \end{aligned} \tag{6}$$

$$\begin{aligned} &\min_{\mathbf{y}} ||\mathbf{Sy}|| && \mathbf{V} \text{ is orthonormal} \\ &\text{s.t. } ||\mathbf{Vy}|| = 1 && \end{aligned} \tag{7}$$

$$\begin{aligned} &\min_{\mathbf{y}} ||\mathbf{Sy}|| && \mathbf{S} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \text{ where } \sigma_1 \geq \sigma_2 \geq \ldots \sigma_n \\ &\text{s.t. } ||\mathbf{y}|| = 1 && \end{aligned} \tag{8}$$

The solution to the problem eq. (8) can be obtained by inspection. Because $\mathbf{S}$ is diagnoal, we first note that

$$\mathbf{Sy} = [\sigma_1 y_1 \quad \sigma_2 y_2 \quad \ldots \quad \sigma_n y_n]^\top \implies ||\mathbf{Sy}||_2 = \sqrt{(\sigma_1 y_1)^2 + (\sigma_2 y_2)^2 + \ldots + (\sigma_n y_n)^2} \tag{9}$$

Note that we can choose the $||\cdot||_2$ norm w.l.o.g (without loss of generality) because the above reformulation of the minimization problem is independent of the specific $||\cdot||_p$ norm.

Now, from the constraint in eq. (8) we know that the total length of the $\mathbf{y}$-vector must be equal to 1. From eq. (9) we see that quantity to be minimized $||\mathbf{Sy}||_2$ is a sum of squares, where the coefficients are decreasing (e.g. σ_n is the smallest). Therefore, by insepction, we find that the optimal solution $\mathbf{y}^*$ is given by

$$\mathbf{y}^* = [0 \quad 0 \quad \ldots \quad 1]^\top$$

as this solution maximizes the contribution of the smallest singular value in eq. (9).

Finally, we re-substitute the solution into eq. (6) and obtain the solution $\mathbf{m}^*$ to the original problem eq. (3):

$$\mathbf{y}^* = \mathbf{V}^\top\mathbf{m}^* \implies \mathbf{m}^* = \mathbf{Vy}^* = \mathbf{V}(:, \text{end})$$

In other words, the optimal solution is given by the last column of the matrix $\mathbf{V}$, as shown in the lecture.