Vision Algorithms for Mobile Robotics

Lecture 11
Tracking

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Lab Exercise – Today

Implement the Kanade-Lucas-Tomasi (KLT) tracker
Template tracking

Goal: follow a template image in a video sequence
Goal: estimate the transformation $W$ (warp) between a template $T$ and the current image $I$
Common 2D Transformations (recall Lecture 03, slides 36-37)

We denote the transformation $W(x, p)$ and $p$ the set of parameters $p = (a_1, a_2, ..., a_n)$

- Translation
  
  $x' = x + a_1$
  $y' = y + a_2$

- Euclidean
  
  $x' = x\cos(a_3) - y\sin(a_3) + a_1$
  $y' = x\sin(a_3) + y\cos(a_3) + a_2$

- Affine
  
  $x' = a_1x + a_3y + a_5$
  $y' = a_2x + a_4y + a_6$

- Projective (homography)
  
  $x' = \frac{a_1x + a_2y + a_3}{a_7x + a_8y + 1}$
  $y' = \frac{a_4x + a_5y + a_6}{a_7x + a_8y + 1}$
Two possible approaches

• Indirect methods (i.e., feature based)
  ✓ Can cope with large frame-to-frame motions (large basin of convergence)
  × Slow due to costly feature extraction, matching, and outlier removal (e.g., RANSAC)

• Direct methods
  ✓ All information in the image can be exploited (higher accuracy, higher robustness to motion blur and weak texture (i.e., weak gradients))
  ✓ Increasing the camera frame-rate reduces computational cost per frame (no RANSAC needed)
  × Very sensitive to initial value → limited frame-to-frame motion (small basin of convergence)
Indirect methods work by detecting and matching features (i.e., feature based)

1. Detect and match features that are invariant to scale, rotation, view point changes (e.g., SIFT)
2. Geometric verification (RANSAC) (e.g., 4-point RANSAC for planar objects, or 5 or 8-point RANSAC for 3D objects)
3. Refine estimate by minimizing the sum of squared reprojection errors between the observed feature $f^i$ in the current image and the warped corresponding feature $W(x^i, p)$ from the template

$$p = \arg\min_p \sum_{i=1}^{N} \| W(x^i, p) - f^i \|^2$$

**Pros**: can cope with large frame-to-frame motion and strong illumination changes

**Cons**: computationally expensive
**Direct methods** work by directly processing pixel intensities (i.e. without features)

Goal: estimate the parameters $\mathbf{p}$ of the transformation $W(\mathbf{x}, \mathbf{p})$ that minimize the Sum of Squared Differences:

$$\mathbf{p} = \arg\min_{\mathbf{p}} \sum_{\mathbf{x} \in T} \left[ I(W(\mathbf{x}, \mathbf{p})) - T(\mathbf{x}) \right]^2$$

*Every yellow dot in this image denotes a pixel*
Assumptions

• **Brightness constancy**
  The intensity of the pixels to track does not change much over consecutive frames → **It does not cope with strong illumination changes**

• **Temporal consistency**
  Small frame-to-frame motion (1-2 pixels).
  → **It does not cope with large frame-to-frame motion.** However, this can be addressed using coarse-to-fine multi-scale implementations (see later)

• **Spatial coherency**
  All pixels in the template undergo the same transformation (i.e., they all lay on the same 3D surface)
  → **No errors in the template image boundaries:** only the object to track appears in the template image
  → **No occlusion:** the entire template is visible in the input image
The Kanade-Lucas-Tomasi (KLT) tracker

- Simplified case: pure translation
- General case


KLT tracking applied to pure translation

Consider the reference patch centered at \((x, y)\) in image \(I_0\) and the shifted patch centered at \((x + u, y + v)\) in image \(I_1\). The patch has size \(\Omega\). We want to find the motion vector \((u, v)\) that minimizes the Sum of Squared Differences (SSD):

\[
SSD(u, v) = \sum_{x, y \in \Omega} \left( I_0(x, y) - I_1(x + u, y + v) \right)^2
\]

\[I_1(x + u, y + v) \approx I_1(x, y) + I_x u + I_y v\]

\[
\Rightarrow SSD(u, v) \approx \sum_{x, y \in \Omega} (I_0(x, y) - I_1(x, y) - I_x u - I_y v)^2
\]

\[
\Rightarrow SSD(u, v) \approx \sum_{x, y \in \Omega} (\Delta I - I_x u - I_y v)^2
\]

This is a simple quadratic function in two variables \((u, v)\)
KLT tracking applied to pure translation

\[ \Rightarrow SSD(u, v) \cong \sum_{x,y \in \Omega} (\Delta I - I_x u - I_y v)^2 \]

To minimize it, we differentiate it with respect to \((u, v)\) and equate it to zero:

\[
\frac{\partial SSD}{\partial u} = 0, \quad \frac{\partial SSD}{\partial v} = 0
\]

\[
\frac{\partial SSD}{\partial u} = 0 \Rightarrow -2 \sum l_x(\Delta I - I_x u - I_y v) = 0
\]

\[
\frac{\partial SSD}{\partial v} = 0 \Rightarrow -2 \sum l_y(\Delta I - I_x u - I_y v) = 0
\]
KLT tracking applied to pure translation

\[ \sum I_x (\Delta I - I_x u - I_y v) = 0 \]

\[ \sum I_y (\Delta I - I_x u - I_y v) = 0 \]

- Linear system of two equations in two unknowns
- We can write them in matrix form:

\[
\begin{bmatrix}
\sum I_x I_x & \sum I_x I_y \\
\sum I_x I_y & \sum I_y I_y
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix} =
\begin{bmatrix}
\sum I_x \Delta I \\
\sum I_y \Delta I
\end{bmatrix}

\Rightarrow
\begin{bmatrix}
u \\
v
\end{bmatrix} =
\begin{bmatrix}
\sum I_x I_x & \sum I_x I_y \\
\sum I_x I_y & \sum I_y I_y
\end{bmatrix}^{-1}
\begin{bmatrix}
\sum I_x \Delta I \\
\sum I_y \Delta I
\end{bmatrix}
\]

Notice that these are NOT matrix products but pixel-wise products!
KLT tracking applied to pure translation

\[ \sum I_x (\Delta I - I_x u - I_y v) = 0 \]

\[ \sum I_y (\Delta I - I_x u - I_y v) = 0 \]

• Linear system of two equations in two unknowns
• We can write them in matrix form:

\[
\begin{bmatrix}
\sum I_x I_x & \sum I_x I_y \\
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\sum I_x I_x & \sum I_x I_y \\
\sum I_x I_y & \sum I_y I_y
\end{bmatrix}^{-1}
\begin{bmatrix}
\sum I_x \Delta I \\
\sum I_y \Delta I
\end{bmatrix}
\]

Haven’t we seen this matrix already?
This is the M matrix of the Harris detector (Lecture 05)
KLT tracking applied to pure translation

For $M$ to be invertible, $\det(M)$ should be non zero, which means that its eigenvalues should be large (i.e., not a flat region, not an edge) → in practice, it should be a corner or more generally contain texture!

$$M = \begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix} = R^{-1} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} R$$

- Edge → $\det(M)$ is low
- Flat → $\det(M)$ is low
- Texture → $\det(M)$ is high
Application to Corner Tracking

Color encodes motion direction

When does it fail?
Application to Optical Flow

What if you track every single pixel in the image?
Application to Optical Flow
Application to Optical Flow
Application to Optical Flow
The Kanade-Lucas-Tomasi (KLT) tracker

• Simplified case: pure translation
• General case


KLT applied to generic warps

Goal: estimate the parameters $\mathbf{p}$ of the transformation $W(\mathbf{x}, \mathbf{p})$ that minimize the SSD:

$$SSD = \sum_{\mathbf{x} \in T} [I(W(\mathbf{x}, \mathbf{p})) - T(\mathbf{x})]^2$$

* Every yellow dot in this image denotes a pixel
KLT applied to generic warps

Goal: estimate the parameters \( p \) of the transformation \( W(x, p) \) that minimize the SSD:

\[
SSD = \sum_{x \in T} \left[ I(W(x, p)) - T(x) \right]^2
\]

• KLT follows the Gauss-Newton method for minimization, that is:
  • Applies a first-order approximation of the warp
  • Attempts to minimize the SSD iteratively
KLT applied to generic warps

\[ SSD = \sum_{x \in T} \left[ I(W(x, p)) - T(x) \right]^2 \]

- Assume that an initial estimate of \( p \) is known. Then, we want to find the increment \( \Delta p \) that minimizes

\[ SSD = \sum_{x \in T} \left[ I(W(x, p + \Delta p)) - T(x) \right]^2 \]

- First-order Taylor approximation of \( I(W(x, p + \Delta p)) \) yields to:

\[ I(W(x, p + \Delta p)) \approx I(W(x, p)) + \nabla I \frac{\partial W}{\partial p} \Delta p \]

\( \nabla I = [I_x, I_y] = \) Image gradient evaluated at \( W(x, p) \)

\( \frac{\partial W}{\partial p} = \) Jacobian of the warp \( W(x, p) \)
KLT applied to generic warps

\[
SSD = \sum_{x \in T} [I(W(x, p + \Delta p)) - T(x)]^2
\]

- By replacing \( I(W(x, p + \Delta p)) \) with its 1st order approximation, we get

\[
SSD = \sum_{x \in T} \left[ I(W(x, p)) + \nabla I \frac{\partial W}{\partial p} \Delta p - T(x) \right]^2
\]

- How do we minimize it?
- We differentiate SSD with respect to \( \Delta p \) and we equate it to zero, i.e., \( \frac{\partial SSD}{\partial \Delta p} = 0 \)
KLT applied to generic warps

$$SSD = \sum_{x \in T} \left[ I(W(x, p)) + \nabla I \frac{\partial W}{\partial p} \Delta p - T(x) \right]^2$$

$$\frac{\partial SSD}{\partial \Delta p} = 2 \sum_{x \in T} \left[ \nabla I \frac{\partial W}{\partial p} \right]^T \left[ I(W(x, p)) + \nabla I \frac{\partial W}{\partial p} \Delta p - T(x) \right]$$

$$\frac{\partial SSD}{\partial \Delta p} = 0$$

$$2 \sum_{x \in T} \left[ \nabla I \frac{\partial W}{\partial p} \right]^T \left[ I(W(x, p)) + \nabla I \frac{\partial W}{\partial p} \Delta p - T(x) \right] = 0 \Rightarrow$$
KLT applied to generic warps

\[ \Delta \mathbf{p} = H^{-1} \sum_{x \in \mathcal{T}} \left[ \nabla I \left( \frac{\partial W}{\partial \mathbf{p}} \right)^\text{T} \right] \left[ T(x) - I(W(x, p)) \right] = \]

\[ H = \sum_{x \in \mathcal{T}} \left[ \nabla I \frac{\partial W}{\partial \mathbf{p}} \right]^\text{T} \left[ \nabla I \frac{\partial W}{\partial \mathbf{p}} \right] \]

Second moment matrix (Hessian) of the warped image

What does $H$ look like when the warp is a pure translation?

Notice that these are NOT matrix products but pixel-wise products!
KLT algorithm: Discussion

KLT algorithm is iterated until convergence by following a predict-correct cycle

1. A prediction $I(W(x, p))$ of the warped image is computed from an initial estimate of $p$

2. The correction parameter $\Delta p$ is then computed as a function of the error $T(x) - I(W(x, p))$ between the prediction and the template. The larger this error, the larger the correction applied.

3. Steps 1 & 2 are iterated until the error is smaller than a threshold and the output parameters are used as input for the next frame.

\[ \Delta p = H^{-1} \sum_{x \in T} [\nabla I \frac{\partial W}{\partial p}^T] [T(x) - I(W(x, p))] \]
KLT algorithm: Discussion

• How to get the initial estimate \( p \)?
• When does the Lucas-Kanade fail?
  • If the initial estimate is too far, then the linear approximation does not longer hold
    → Solution: Coarse-to-fine implementations (see next slide)
  • Too poor texture
    → Solution: increase the aperture (see next slide)
  • Deviations from mathematical warp model: object deformations, illumination changes, etc.
    → Solution: Update the template with the last image: problem: drift
• Occlusions
• Template background
Coarse-to-fine estimation

Pyramid of image I

Pyramid of image T

$u = 1.25$ pixels

$u = 2.5$ pixels

$u = 5$ pixels

$u = 10$ pixels

of motion displacement
Aperture Problem

• Consider the motion of the following corner
Aperture Problem

- Consider the motion of the following corner
Aperture Problem

• Now, look at the local brightness changes through a small aperture
Aperture Problem

• Now, look at the local brightness changes through a small aperture
Aperture Problem

• Now, look at the local brightness changes through a small aperture
• We cannot always determine the motion direction → Infinite motion solutions may exist!
• Solution?
Aperture Problem

• Now, look at the local brightness changes through a small aperture
• We cannot always determine the motion direction → Infinite motion solutions may exist!
• Solution?
  • Increase aperture size!
Generalization of KLT

• The same concept (predict/correct) can be applied to tracking of 3D object (in this case, what is the transformation to estimate? What is the template?)
Generalization of KLT

• The same concept (predict/correct) can be applied to tracking of 3D object (in this case, what is the transformation to estimate? What is the template?)

• In order to deal with wrong prediction, it can be implemented in a Particle-Filter fashion (using multiple hypotheses that need to be validated)
Math Refresher
Common 2D Transformations in Matrix form

We denote the transformation $W(\mathbf{x}, \mathbf{p})$ and $\mathbf{p}$ the set of parameters $p = (a_1, a_2, \ldots, a_n)$

- **Translation**
  $$W(\mathbf{x}, \mathbf{p}) = \begin{bmatrix} x + a_1 \\ y + a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Euclidean**
  $$W(\mathbf{x}, \mathbf{p}) = \begin{bmatrix} x\cos(a_3) - y\sin(a_3) + a_1 \\ x\sin(a_3) + y\cos(a_3) + a_2 \end{bmatrix} = \begin{bmatrix} \cos(a_3) & -\sin(a_3) & a_1 \\ \sin(a_3) & \cos(a_3) & a_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Affine**
  $$W(\mathbf{x}, \mathbf{p}) = \begin{bmatrix} a_1x + a_3y + a_5 \\ a_2x + a_4y + a_6 \end{bmatrix} = \begin{bmatrix} a_1 & a_3 & a_5 \\ a_2 & a_4 & a_6 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **Projective (homography)**
  $$W(\bar{\mathbf{x}}, \mathbf{p}) = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
# Common 2D Transformations in Matrix form

<table>
<thead>
<tr>
<th>Name</th>
<th>Matrix</th>
<th># D.O.F.</th>
<th>Preserves:</th>
<th>Icon</th>
</tr>
</thead>
</table>
| translation        | \[
\begin{bmatrix}
    1 & 0 & t_x \\
    0 & 1 & t_y \\
    0 & 0 & 1
\end{bmatrix}
\] | 2        | orientation + ···          |      |
| rigid (Euclidean)  | \[
\begin{bmatrix}
    R & t \\
    0 & 1
\end{bmatrix}
\] | 3        | lengths + ···              |      |
| similarity         | \[
\begin{bmatrix}
    sR & t \\
    0 & 1
\end{bmatrix}
\] | 4        | angles + ···               |      |
| affine             | \[
\begin{bmatrix}
    A \\
    0
\end{bmatrix}
\] | 6        | parallelism + ···          |      |
| projective         | \[
\begin{bmatrix}
    H \\
    0
\end{bmatrix}
\] | 8        | straight lines             |      |

\[
W(x, p) = \begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

\[
W(x, p) = \begin{bmatrix}
\cos(a_3) & -\sin(a_3) & a_1 \\
\sin(a_3) & \cos(a_3) & a_2 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

\[
W(x, p) = \begin{bmatrix}
\cos(a_3) & -\sin(a_3) & a_1 \\
\sin(a_3) & \cos(a_3) & a_2 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

\[
W(x, p) = \begin{bmatrix} a_1 & a_3 & a_5 \\ a_2 & a_4 & a_6 \\ a_7 & a_8 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

\[
W(x, p) = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]
Derivative and gradient

• Function: $f(x)$

• Derivative: $f'(x) = \frac{df}{dx}$, where $x$ is a scalar

• Function: $f(x_1, x_2, ..., x_n)$

• Gradient: $\nabla f(x_1, x_2, ..., x_n) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right)$
Jacobian

- \( F(x_1, x_2, \ldots, x_n) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix} \) is a vector-valued function

- The derivative in this case is called Jacobian \( \frac{\partial F}{\partial x} \):

\[
\frac{\partial F}{\partial x} = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_1}{\partial x_n} \\
\vdots \\
\frac{\partial f_m}{\partial x_1}, \ldots, \frac{\partial f_m}{\partial x_n}
\end{bmatrix}
\]

Carl Gustav Jacob (1804-1851)
Displacement-model Jacobians $\nabla W_p$

\[ \mathbf{p} = (a_1, a_2, ..., a_n) \]

- **Translation:**
  \[ W(x, \mathbf{p}) = \begin{bmatrix} x + a_1 \\ y + a_2 \end{bmatrix} \quad \frac{\partial W}{\partial \mathbf{p}} = \begin{bmatrix} \frac{\partial W_1}{\partial a_1} & \frac{\partial W_1}{\partial a_2} \\ \frac{\partial W_2}{\partial a_1} & \frac{\partial W_2}{\partial a_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

- **Euclidean:**
  \[ W(x, \mathbf{p}) = \begin{bmatrix} x\cos(a_3) - y\sin(a_3) + a_1 \\ x\sin(a_3) + y\cos(a_3) + a_2 \end{bmatrix} \quad \frac{\partial W}{\partial \mathbf{p}} = \begin{bmatrix} 1 & 0 & -x\sin(a_3) - y\cos(a_3) \\ 0 & 1 & x\cos(a_3) - y\sin(a_3) \end{bmatrix} \]

- **Affine:**
  \[ W(x, \mathbf{p}) = \begin{bmatrix} a_1x + a_3y + a_5 \\ a_2x + a_4y + a_6 \end{bmatrix} \quad \frac{\partial W}{\partial \mathbf{p}} = \begin{bmatrix} x & 0 & y & 0 & 1 & 0 \\ 0 & x & 0 & y & 0 & 1 \end{bmatrix} \]
Readings

Understanding Check

Are you able to answer the following questions?

• What is the problem formulation of tracking?
• Difference between direct and indirect methods and their pros and cons
• Can you illustrate tracking methods using point features?
• Are you able to explain the underlying assumptions behind direct methods, derive their mathematical expression for the case of pure rotation and the meaning of the M matrix?
• When is the M matrix invertible and when not?
• What is optical flow?
• Are you able to describe the working principle of KLT for a generic warp?
• What functional does KLT minimize?
• What is the Hessian matrix and for which warping function does it coincide to that used for pure translation?
• Can you list Lukas-Kanade failure cases and how to overcome them?
• How do we get the initial guess?
• Can you illustrate the coarse-to-fine Lucas-Kanade implementation?
• What is the aperture problem and how can we overcome it?