Notes on Convolution

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1 Notation

In discrete time, consider two signals $f[i]$, $g[i]$. Their convolution $*$ is defined as

$$(f * g)[n] = \sum_{m=-M}^{m=M} f[n-m]g[m]$$

with support $\{-M, -M+1, \ldots, M-1, M\}$. Outside of the support, the signal is zero.

In continuous time, consider two functions $f(x)$, $g(x)$. Their convolution is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

2 Properties

Commutation

$$f * g = g * f$$ (1)

Proof (through a change of variables):

$$(f * g)(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du = \int_{-\infty}^{\infty} f(x-v)g(v)dv = \int_{-\infty}^{\infty} g(v)f(x-v)dv = (g * f)(x)$$

Dirac Delta

Let $\delta(x)$ denote the dirac delta function, defined as

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \text{undefined} & x = 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(u)du = 1$$

$$f * \delta = f$$ (2)

Proof (through using the definition):

$$(f * \delta)(x) = \int_{-\infty}^{\infty} f(u)\delta(x-u)du = f(x)$$

Derivatives

$$(f * g)' = f' * g = \overline{g'} * f$$ (3)

Proof:

$$(f * g)' = \frac{d}{dx} \int_{-\infty}^{\infty} f(u)g(x-u)du = \int_{-\infty}^{\infty} f(u) \frac{d}{dx}g(x-u)du = f * g'$$