1 Proof

Nullspace

In a DLT problem with $N$ points, we need to find the solution the following system of equations (and ignore the trivial solution $m = 0$):

$$Qm = 0, \quad \text{where} \quad Q \in \mathbb{R}^{2N \times 12}, \quad m \in \mathbb{R}^{12}, \quad m \neq 0$$

(1)

Mathematically, this means that the vector $m$ has to lie in the nullspace of $Q$:

$$Qm = 0 \iff m \in \text{Null}(Q)$$

Note that, for any $m \in \text{Null}(Q)$, any $\alpha$-multiple also lies in the nullspace, i.e.

$$Qm = 0 \implies Q\alpha m = 0, \quad \alpha \in \mathbb{R}$$

(2)

Minimization Problem

Using the fact that (any) norm of a vector is greater or equal to zero, we can reformulate eq. (1) as a minimization problem as follows:

$$\min_m ||Qm|| \quad \text{s.t.} \quad ||m|| = 1$$

(3)

The constraint on the norm of $||m||$ ensures that the trivial solution is ignored. Note that it does not change the solution, because of the scale-invariance of the solution, see eq. (2).

If a solution to eq. (1) exists, the minimization above will find it. If no (exact) solution exists, the minimization problem will find the best possible solution, in the sense of the norm. If we chose the norm $||\cdot||$ to be the L2-norm, we obtain a solution that is optimal in the least-squares sense.
Solution to the minimization

To solve the minimization problem eq. (3), we use the properties of the singular value decomposition to simplify the problem and then solve it by inspection.

\[
\begin{align*}
\min_{m} & \quad \|Qm\| \\
\text{s.t.} & \quad \|m\| = 1
\end{align*}
\]

let \( Q = USV^T \)

with \( U \in \mathbb{R}^{2N \times 12} \), \( S \in \mathbb{R}^{12 \times 12} \), \( V \in \mathbb{R}^{12 \times 12} \) \hspace{1cm} (4)

\[
\begin{align*}
\min_{m} & \quad \|USV^Tm\| \\
\text{s.t.} & \quad \|m\| = 1
\end{align*}
\]

\( U \) is orthonormal (no change to length of vector) \hspace{1cm} (5)

\[
\begin{align*}
\min_{m} & \quad \|SV^Tm\| \\
\text{s.t.} & \quad \|m\| = 1
\end{align*}
\]

let \( y = V^Tm \)

\( Vy = VV^Tm = m \) because \( V \) is orthonormal \hspace{1cm} (6)

\[
\begin{align*}
\min_{y} & \quad \|Sy\| \\
\text{s.t.} & \quad \|Vy\| = 1
\end{align*}
\]

\( V \) is orthonormal \hspace{1cm} (7)

\[
\begin{align*}
\min_{y} & \quad \|Sy\| \\
\text{s.t.} & \quad \|y\| = 1
\end{align*}
\]

\( S = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \) where \( \sigma_1 \geq \sigma_2 \geq \ldots \sigma_n \) \hspace{1cm} (8)

The solution to the problem eq. (8) can be obtained by inspection. Because \( S \) is diagonal, we first note that

\[
Sy = [\sigma_1y_1 \quad \sigma_2y_2 \quad \ldots \quad \sigma_ny_n]^T \implies \|Sy\|_2 = \sqrt{(\sigma_1y_1)^2 + (\sigma_2y_2)^2 + \ldots + (\sigma_ny_n)^2} \hspace{1cm} (9)
\]

Note that we can choose the \( \| \cdot \|_2 \) norm w.l.o.g (without loss of generality) because the above reformulation of the minimization problem is independent of the specific \( \| \cdot \|_p \) norm.

Now, from the constraint in eq. (8) we know that the total length of the \( y \)-vector must be equal to 1. From eq. (9) we see that quantity to be minimized \( \|Sy\|_2 \) is a sum of squares, where the coefficients are decreasing (e.g. \( \sigma_n \) is the smallest). Therefore, by inspection, we find that the optimal solution \( y^* \) is given by

\[
y^* = [0 \quad 0 \quad \ldots \quad 1]^T
\]

as this solution maximizes the contribution of the smallest singular value in eq. (9).

Finally, we re-substitute the solution into eq. (6) and obtain the solution \( m^* \) to the original problem eq. (3):

\[
y^* = V^Tm^* \implies m^* = Vy^* = V(:, \text{end})
\]

In other words, the optimal solution is given by the last column of the matrix \( V \), as shown in the lecture.