SPACE RESECTION : FAILURE CASES

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Abstract

It is well known that there exists a "danger cylinder" for space resection, analogous to the "danger circle" in the plane, on which the resection is unstable. This paper shows how the existence of this surface may be detected and draws attention to the way in which the instability differs from that in the plane.

IN his very interesting paper on space resection Mr. A. D. N. Smith (1965, p. 118)[1] draws attention to certain failure cases, which he regards as cases in which the solutions of the equations are algebraically indeterminate. This is, however, to omit cases in which the solution is not strictly indeterminate but in which it is nevertheless unstable, and thus as dangerous as if it were indeterminate. Moreover, these cases are less trivial in the present context in the sense that they can only too easily occur in practice and are one cause, at least, for regarding space resection of the single picture as unsatisfactory for the determination of the elements of outer orientation. Although the problem is old and no new results are given in this paper, recent and not so recent textbooks merely refer to the instability without deriving it and investigating its nature, and an accessible treatment may be useful. Moreover, there is a natural tendency to draw too close an analogy with the unstable case of plane resection and the opportunity is taken here to point out the essential differences.

We start with an investigation of the general problem of the instability of the solutions of equations which may be non-linear. Suppose we wish to find n unknowns $x_1, x_2, ..., x_n$ by the solution of n equations of the form,

$$f_i(x_1, x_2, ..., x_n) = 0 \quad (i = 1, 2, ..., n).$$

For a solution to be stable we must not be able to make first-order changes in any of the unknowns such that the equations remain satisfied. That is to say, the equations

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\[2mm] \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\[2mm] \cdots & \cdots & \cdots & \cdots \\[2mm] \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} dx_1 \\[2mm] dx_2 \\[2mm] \vdots \\[2mm] dx_n \end{pmatrix} = \begin{pmatrix} 0 \\[2mm] 0 \\[2mm] \vdots \\[2mm] 0 \end{pmatrix} \quad \quad ....(1)$$

must have no solutions for $dx_1, dx_2, ..., dx_n$ other than zeros. The condition for this is simply that the determinant of the matrix of the partial differential coefficients

must not vanish; that is, the matrix must be non-singular. The *determinant* is known as the *Jacobian*, and we will call the matrix the *Jacobian matrix*. Unstable solutions will arise when the Jacobian matrix is singular and to disclose these we have only to equate the Jacobian to zero and investigate the consequences. The character of the instability will depend upon the rank of the Jacobian matrix. For example, if the rank is $n-1$, then $dx_1, dx_2, ..., dx_n$ are uniquely determined apart from an arbitrary common factor. We shall see the significance of this below.

Regarding u, v, w in Mr. Smith's equations (2), (3), (4) as functions of x, y, z, the coordinates of the vertex, the Jacobian can be verified as being

$$\frac{1}{uvw} \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{vmatrix} \begin{vmatrix} 0 & v-w\cos\alpha & w-v\cos\alpha \\ u-w\cos\beta & 0 & w-u\cos\beta \\ u-v\cos\gamma & v-u\cos\gamma & 0 \end{vmatrix},$$

where $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are respectively the coordinates of the three controls.

The left-hand determinant is proportional to the volume of a tetrahedron of which S and the controls, A, B, C are vertices. If it vanishes, the Jacobian vanishes, the volume of the tetrahedron is zero and the points S, A, B, C are coplanar. This will always be the case if A, B, C are collinear (the indeterminate case cited by Mr. Smith), but it will also be the case if S is in the plane of A, B, C. In this last case there will be instability but not, in general, indeterminancy. If we express $u, v, w, \cos\alpha, \cos\beta, \cos\gamma$ in terms of the coordinates of S, A, B, C and equate the right-hand determinant to zero we will, after some tedious algebra, obtain an equation in the coordinates of S that represents the equation of a circular cylinder circumscribing A, B, C with axis normal to the plane of A, B, C. The situation is shown in Fig. 1.
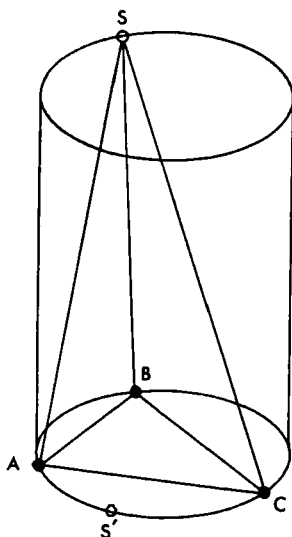

FIG. 1

This cylinder is known as the *danger cylinder* in the literature, but too close an analogy with the *danger circle* of plane resection should not be made. If the vertex lies on the danger cylinder at S, not in the plane of A, B, C, there is instability but not

indeterminancy; and the reader can easily convince himself that any small displacement of S *on the surface* of the cylinder must change at least one of the apex angles to a first order. Of course, if the vertex is at S', say, in the plane of A, B, C, there will be no first-order change of apex angles for displacements on the cylinder; this being, in effect, the plane resection case extended to allow for small displacements out of the plane. It is, in fact, one of the misleading characteristics of the failure case in plane resection that the instability is along the circumference of the danger circle *on which* instability occurs. This is a very special case, for in general the instability will be in a direction that takes the point off the curve or surface on which instability occurs. For this reason instability is usually a first-order effect, which it is not in the plane case when we can move as far as we like along the danger circle without recovering stability. The instability in space resection must therefore be, in general, in a direction that takes the vertex off the danger cylinder and we ask ourselves what this direction can be.
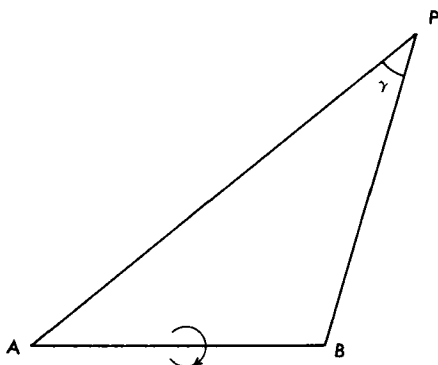

FIG. 2

Consider the locus of a point P (Fig. 2) such that the chord AB (Fig. 1) subtends a fixed angle γ at P. If we fix the plane PAB then the locus of P is a circle with AB as chord. But this circle may be rotated about AB as axis without changing γ. Hence the locus of P is the *toroid* obtained by rotating a circle about an axis AB cutting the circle. The vertex S can thus be regarded as the point (or points) of intersection of three toroids having AB, AC, BC respectively as axes and γ, β, α respectively as subtended angles. The points at which there is instability are simply those at which the surfaces do not "really" intersect, i.e. points at which the three surfaces have a common tangent or a common tangent plane, the instability being in a direction along the common tangent or in the common tangent plane. It is clear that for any point S' on the circle circumscribing A, B, C, the three toroids have a common tangent plane, and that this tangent plane is also tangent to the cylinder.

It is instructive to approach the problem geometrically in this way. Mr. Smith's equations (2), (3), (4), regarded as equations in x, y, z, are the equations of the three toroids whose intersections we are seeking. Let us designate them by

$$f_i(x \, y \, z) = 0 \quad (i = 1, 2, 3).$$

If x_0, y_0, z_0 is a point on the surface and if $x_0 + dx, y_0 + dy, z_0 + dz$ is an adjacent point, then

$$f_i(x_0 + dx, y_0 + dy, z_0 + dz) = 0.$$

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Expanding by Taylor's theorem and putting $f_i(x_0\, y_0\, z_0) = 0$, we have

$$\left(\frac{\partial f_i}{\partial x}\right)_0 dx + \left(\frac{\partial f_i}{\partial y}\right)_0 dy + \left(\frac{\partial f_i}{\partial z}\right)_0 dz = 0,$$

where the suffix 0 indicates that the partial differential coefficients are to be calculated at x_0, y_0, z_0. The vector

$$\left[\left(\frac{\partial f_i}{\partial x}\right)_0 \left(\frac{\partial f_i}{\partial y}\right)_0 \left(\frac{\partial f_i}{\partial z}\right)_0\right]$$

is thus perpendicular to $[dx\, dy\, dz]$ which is a vector lying in the surface. Hence the vector of the partial differential coefficients is normal to the surface at x_0, y_0, z_0.

Let $(x\, y\, z)$ be a point in space such that

$$\left(\frac{\partial f_i}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f_i}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial f_i}{\partial z}\right)_0 (z - z_0) = 0.$$

The vector $[(x - x_0)(y - y_0)(z - z_0)]$ is then perpendicular to the normal to the surface at $(x_0\, y_0\, z_0)$ and the equation thus represents a plane parallel to the tangent plane at $(x_0\, y_0\, z_0)$. Moreover, $x = x_0, y = y_0, z = z_0$ satisfy the equation and hence it *is* the tangent plane at $(x_0\, y_0\, z_0)$.

Consider now the three toroids intersecting at $(x_0\, y_0\, z_0)$: any points common to the three tangent planes will be solutions of

$$\begin{pmatrix} \left(\frac{\partial f_1}{\partial x}\right)_0 & \left(\frac{\partial f_1}{dy}\right)_0 & \left(\frac{\partial f_1}{\partial z}\right)_0 \\ \left(\frac{\partial f_2}{\partial x}\right)_0 & \left(\frac{\partial f_2}{\partial y}\right)_0 & \left(\frac{\partial f_2}{\partial z}\right)_0 \\ \left(\frac{\partial f_3}{\partial x}\right)_0 & \left(\frac{\partial f_3}{\partial y}\right)_0 & \left(\frac{\partial f_3}{\partial z}\right)_0 \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \ldots\ldots(2)$$

In general, there will be only one such point, viz. $x = x_0$, $y = y_0$ and $z = z_0$. If, however, the square matrix is singular with rank two $x - x_0$, $y - y_0$ and $z - z_0$ will be uniquely determined apart from an arbitrary common factor. There will thus be a line of points satisfying all three equations and this line will be the common tangent to the three surfaces at x_0, y_0, z_0. If the matrix has rank one there is effectively only one equation to be satisfied by x, y, z and this equation will represent the common tangent plane at x_0, y_0, z_0.

We note, however, that the square matrix in (2) is precisely the Jacobian matrix in (1) which we considered first in this paper; and we thus see that the purely geometrical ideas of common tangents and common tangent planes lead to the same conclusions as before.

REFERENCE

1. SMITH, A. D. N., "The Explicit Solution of the Single Picture Resection Problem, with a Least Squares Adjustment to Redundant Control", *Photogrammetric Record*, Vol. V, No. 26, October 1965.

Résumé

L'existence du cylindre dit "de danger" est bien connue dans la théorie de relèvement dans l'espace. Si le centre de projection se trouve sur ce cylindre la solution du problème est indéfinie. Cette communication expose la base mathématique du problème et met en évidence les différences entre le problème dans l'espace et le problème analogue dans le plan.

ANNUAL GENERAL MEETING, 1965

THE Annual General Meeting of the Photogrammetric Society was held at Burlington House, London, on 23rd November, 1965. There were forty-nine members present.

The Minutes of the 1964 meeting were approved and signed by the President, Mr. J. E. Odle, who then gave his Presidential Address, which is printed below.

The Honorary Treasurer presented his Report, in which he referred to the considerable reduction in the production cost of *The Photogrammetric Record*.

The nominated Officers and Council of the Society for the year 1965–6 were declared elected, none of the posts being contested. The Honorary Auditor was re-elected.

The meeting approved the election of Dr. W. Schermerhorn and Major General R. Ll. Brown as Honorary Members of the Society, which was proposed by the President as follows:

"Dr. Schermerhorn is not only a world figure in photogrammetry but has the unique distinction of being the only former Prime Minister of his country in our Society. His contribution to the International Society for Photogrammetry has been immense.

The International Training Centre for Aerial Survey at Delft was his brain-child and achieved its international reputation as the major photogrammetric educational establishment through his energy and skill."

"As President of our Society and the International Society for Photogrammetry General Brown played a leading part in the acceptance of responsibility for the London Congress in 1960 when his leadership for the period of preparation and the Congress itself was a major contribution. As the first Vice-President to hold office in the International Society for Photogrammetry he continued actively working for the International Society for Photogrammetry until the Lisbon Congress in 1964.

"Already a medallist it is a fitting further recognition of his work that he is now appointed as Honorary Member."

THE PRESIDENT'S MEDAL

The President's Medal for 1965 was presented to Mr. R. W. Fish by the President, after he had read the Citation. Mr. Fish made a brief speech of thanks and then delivered his paper "Navigational and Instrument Aids to Air Survey" which appears in this issue.

CITATION

Ronald William Fish graduated from the Imperial College of Science in Physics in 1937, and after completing an M.Sc. course in technical optics undertook research work on electron microscopy.

On the outbreak of war he joined Kelvin–Hughes Ltd. (then Henry Hughes Ltd.) as optical designer and research engineer for navigation equipments and in particular

he specialised on astro-navigation instruments for both marine and aircraft use. He was particularly interested in, and responsible for, the development of periscopic sextants, drift sights, gun sights and compasses.

Navigation and reconnaissance were the keystones on which the air operations of the last war rested and as in so many other fields it was the functional character of British equipment which enabled rapid expansion of operational resources.

In 1949 Fish joined the then Ministry of Supply at R.A.E., Farnborough, in the Air Photography Division, when a major programme of camera redesign was initiated. Fish took charge of the section dealing with lens design and the development of cameras for air survey and day reconnaissance under the leadership of Gerald Brock. His contributions quickly became significant. As a physicist he was interested in the mathematical problems of lens design and film resolution, and his work in the theoretical area was recognised as sound and progressive. His great interest, however, has been in the practical application of sound theory to the production of new equipment, and his ability to work from ideal theoretical principles to the practical end of improving the breed while retaining functional simplicity has been evident.

Many engineering features which appear in the range of high- and low-level reconnaissance and survey cameras developed while he was at Farnborough owe their existence to his critical analysis of the problem and his willingness to work with designers and engineers in industry who appreciated his practical approach.

Fish has published a particularly extensive range of departmental reports, but unfortunately many of these were classified as restricted, but have been of the greatest value to those who have had access to them. He has contributed numerous papers to *The Photogrammetric Record* on lens and film resolution, gyro-stabilised mountings and anti-vibration mountings for air cameras. His theoretical work on problems of image quality and image location led to his developing a theory of mountings from which sprang the design and production of a whole new range of isolators in replacement of the historic rubber mountings. He has made many contributions to our knowledge about camera calibration, image compensation and the development of high-resolution systems.

In 1957 Fish was for a time on the staff of the Chief Scientific Adviser to the Ministry of Defence, and in 1960 he returned to the Ministry of Aviation as Assistant Director in the Navigation and Reconnaissance Branch, dealing with automatic flight control systems, weapon aiming, instrument displays, and the like. Early in this period he played a highly important role in stimulating a new approach on survey camera development.

Fish was a member of the Council of the Society from 1956 to 1963 and was Associate President of Commission I in the period 1962 to 1964 when he made a considerable contribution to the Lisbon Congress. In 1964 he was appointed President of Commission I where his energy and interest are being devoted to the London Symposium planned for 1966. He has been a regular and useful contributor to our Brains Trusts and in discussions at our Technical Proceedings.

Fish is a man who never hesitates to speak his mind and consequently his friends and associates can always expect candid criticism, but this criticism is invariably constructive, and as an administrator he is keenly aware of the importance of stimulating individual effort towards the solution of difficult problems.

Photogrammetry is fortunate to have a man of his intellectual and practical capacity in its ranks and one with an outstanding record of real achievement. It is a very great pleasure to present to Ronald Fish tonight the highest honour our Society can bestow.

THE PRESIDENT'S ADDRESS

I am pleased to report that our Society continues to make steady progress, membership having risen to 540 as compared with 504 last year. Our financial position is sound in that, after exceptional expenditures arising from participation in the Lisbon Congress, we have nevertheless maintained a small surplus. *The Record*, under the editorship of Professor Thompson, ably assisted by Mr. Atkinson as Assistant Editor, has continued its high standard.

The Conversazione was again well supported, and we were particularly honoured to receive amongst our guests Sir Dudley Stamp, President of the Royal Geographical Society. Our thanks go again to the Royal Institution of Chartered Surveyors for allowing us to use their building for this function.

The Technical Sessions were well attended, and as members will recall we were very pleased to have a lecture from Professor Hallert, who was here with Mr. Johansson to discuss the work of Commission I with Mr. Fish and Mr. Cooper, President and Secretary of the Commission. This February we are to have the honour of a lecture by Dr. Schermerhorn, and I am sure we shall have a large gathering to greet so distinguished a photogrammetrist.

The plans of Commission I for a Symposium in London at University College in September 1966, mentioned in my address last year, are advancing, and we hope this function will be well supported by members.

Work has continued on the glossary of photogrammetric terms, under the leadership of Brigadier Denison. It is now clear that his is a lengthy task, and more time is needed for completion.

We have maintained close links with our kindred societies who are most helpful in their cooperation with us and we again express our thanks to the Geological Society for the use of this building for our Council and General Meetings. We were approached by the British Cartographic Society to arrange a joint paper on "Radar Photography", but this fell through owing to the inability of the lecturer to deliver his paper. There are clearly close links between cartography and photogrammetry, and these will increase with the development of orthophotographic processes. We also took part in a Brains Trust organised by the Institute of Incorporated Photographers at their Eastbourne Conference in May.

Members are reminded of the Annual Essay Prize, submissions for which should be made before the 31st January each year.

Turning now to the work of the Council and Officers, I wish to first pay a special tribute to Mr. Horder, our Honorary Secretary for two and half years who has had to resign as he is now overseas. He has worked very hard for the Society and kept our affairs in such excellent order that when Mr. Lamboit kindly offered to take over temporarily there was virtually no disorganisation.

Major General Edge has resigned from the Council on appointment as Director-General of Ordnance Survey and we shall all miss his calm and shrewd counsel. Mr. Wiggins and Dr. Robbins feel they should now retire from the Council and we thank them for their interest and contributions to the Society's affairs.

I have been greatly supported in my period of office by the Committee Chairmen, Officers and Council Members of the Society and I thank them all for their work, so well supported by their organisations, which has enabled us to function effectively.