

# Filtering & Edge detection

## 1 Numerical Exercises

1. Consider the following 1D image pixel arrays A and B

$$A = [3, 1, 2, 1] \quad B = [7, 7, 6, 4]$$

- (a) Compute the filter F, which was applied as **convolution** to the pixel array A resulting in the output B. Assume that zero padding was applied.

$$B = A * F$$

**Solution:**

$$F = [1, 2, 1]$$

- (b) Compute the filter F, which was applied as **cross-correlation** to the pixel array A resulting in the output B. Assume that zero padding was applied.

$$B = A \otimes F$$

**Solution:**

$$F = [1, 2, 1]$$

- (c) Compute the convolution signal C between the pixel array A and B. Use “reflect across edge” padding. The output should have the same size as the input signal.

**Solution:**

To achieve an output with the same size as the input arrays, the padding can be applied either to A or to B depending on which array is the filter. Another important observation is that both arrays have an even number of entries, thus there exists an ambiguity on which side the output is cropped. This ambiguity should be considered while using standard programming libraries. We provide the solution with the uncropped signal.

If B is the filter, the uncropped solution is:

$$C = [50, 51, 39, 30, 35] \quad \text{with the padded signal} \quad A = [1, 3, 3, 1, 2, 1, 1, 2]$$

If A is the filter, the uncropped solution array is:

$$C = [49, 46, 39, 35, 36] \quad \text{with the padded signal} \quad B = [7, 7, 7, 7, 6, 4, 4, 6]$$

2. What is the convolution output if filter F is applied to the following A matrix assuming zero padding?

$$F = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

**Solution:**

$$F * A = \begin{bmatrix} 1 & -3 & -3 & -3 \\ 7 & 1 & 4 & 2 \\ 14 & 10 & 11 & 3 \\ 11 & 9 & 7 & 1 \end{bmatrix}$$

3. Find the two 1D separable filters  $a, b \in \mathbb{R}^{3 \times 1}$  resulting in the following 2D filters such that:

$$A = ab^T$$

(a)  $A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

**Solution:**

$$a = [0, 1, 0] \quad b = [2, 2, 2]$$

(b)  $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**Solution:**

$$a = [1, 0, 0] \quad b = [0, 0, 1]$$

(c)  $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

**Solution:**

$$a = [2, 1, 2] \quad b = [1, 2, 1]$$

4. In the lecture you've seen the Prewitt filter which calculates a partial derivative. For example, the filter

$$G_x = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

calculates the first partial derivative in the x direction. It approximates  $\frac{\partial I}{\partial x}$ . In a similar fashion, derive a filter  $G_{xx}$  which approximates the second order partial derivative in x  $\frac{\partial^2 I}{\partial x^2}$ . The filter should be of size 3x3.

- (a) Derive the filter  $G_{xx,c}$  using central differences (i.e. one pixel on either side of the current pixel)

**Solution:**

To derive this filter, we can consider a 1D function  $f(x)$  as our desired filter computes a derivative only in one direction. The task is now simplified and we need to approximate  $f''(x)$  using a central difference scheme with step size  $\Delta x$ , i.e. calculate an estimate of  $f''(x)$  using the function values at the positions  $f(x - \Delta x)$ ,  $f(x)$ ,  $f(x + \Delta x)$ . A Taylor expansion of  $f(x)$  around the points  $x$  evaluated at  $x + \Delta x$  and  $x - \Delta x$  yields

$$\begin{aligned} f(x + \Delta x) &= f(x) + \Delta x f'(x) + \Delta x^2 \frac{f''(x)}{2} + \Delta x^3 \frac{f'''(x)}{6} + \Delta x^4 \frac{f^{(4)}(x)}{24} + \text{H.O.T} \\ f(x - \Delta x) &= f(x) - \Delta x f'(x) + \Delta x^2 \frac{f''(x)}{2} - \Delta x^3 \frac{f'''(x)}{6} + \Delta x^4 \frac{f^{(4)}(x)}{24} + \text{H.O.T} \end{aligned}$$

where H.O.T. denotes higher-order terms. Since we are dealing with pixels here, our discretization step size  $\Delta x$  is equal to one. Therefore, we obtain

$$f(x+1) = f(x) + f'(x) + \frac{f''(x)}{2} + \frac{f'''(x)}{6} + \frac{f''''(x)}{24} + \text{H.O.T}$$

$$f(x-1) = f(x) - f'(x) + \frac{f''(x)}{2} - \frac{f'''(x)}{6} + \frac{f''''(x)}{24} + \text{H.O.T}$$

To approximate the second derivative we can sum up the two terms above and solve for  $f''(x)$

$$f(x+1) + f(x-1) - 2f(x) = f''(x) + \frac{f''''(x)}{12} + \text{H.O.T}$$

Since only an approximation is desired, we can discard the higher-order terms of order  $\geq 3$  and get the central second order finite difference

$$f''(x) \approx f(x-1) - 2f(x) + f(x+1)$$

In matrix notation, we can write

$$G_{xx,c} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

- (b) Derive the filter  $G_{xx,f}$  using forward differences (i.e. using only information to the right side of the current pixel)

**Solution:**

Similarly to the central difference case, we can write the Taylor approximations for the neighboring pixels. Consider the points  $x$ ,  $x + \Delta x$ ,  $x + 2\Delta x$ ,  $x + 3\Delta x$ . Following the same procedure as above, we can find that the second order forward difference can be written as

$$f''(x) = 2f(x) - 5f(x+1) + 4f(x+2) - f(x+3)$$

In matrix notation we get

$$G_{xx,f} = \begin{bmatrix} 2 & -5 & 4 & -1 \\ 2 & -5 & 4 & -1 \\ 2 & -5 & 4 & -1 \\ 2 & -5 & 4 & -1 \end{bmatrix}$$