Regularity and Stability Properties of Selective SSMs with Discontinuous Gating

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Abstract

Deep Selective State-Space Models (SSMs), characterized by input-dependent, time-varying parameters, offer significant expressive power but pose challenges for stability analysis, especially with discontinuous gating signals. In this paper, we investigate the stability and regularity properties of continuous-time selective SSMs through the lens of passivity and Input-to-State Stability (ISS). We establish that intrinsic energy dissipation guarantees exponential forgetting of past states. Crucially, we prove that the unforced system dynamics possess an underlying minimal quadratic energy function whose defining matrix exhibits robust AUC_{loc} regularity, accommodating discontinuous gating. Furthermore, assuming a universal quadratic storage function ensures passivity across all inputs, we derive parametric LMI conditions and kernel constraints that limit gating mechanisms, formalizing "irreversible forgetting" of recurrent models. Finally, we provide sufficient conditions for global ISS, linking uniform local dissipativity to overall system robustness. Our findings offer a rigorous framework for understanding and designing stable and reliable deep selective SSMs.

1 Introduction

The pursuit of stable, robust, and predictable behavior in dynamical systems is a cornerstone of scientific and engineering disciplines, with increasing relevance in the era of complex computational models like Neural ODEs [3] and Deep State-Space Models (SSMs) [11, 9]. Foundational paradigms of energy exchange, rooted in physics and later abstracted into powerful mathematical frameworks such as Willems' theory of dissipative systems [29] and Input-to-State Stability (ISS) [25, 14], provide important tools for this pursuit. These energy-based and robustness-centric perspectives allow for rigorous analysis of a system's interaction with its environment and its inherent stability characteristics. Extending far beyond their physical origins, these principles are now crucial for understanding the conditions under which modern AI architectures operate reliably, maintain numerical stability, and process information effectively despite internal complexities and external disturbances. This paper leverages these well-established principles to investigate the regularity and stability properties of continuous-time deep selective SSMs. These models, whose parameters are dynamically modulated by their inputs, pose distinct analytical questions at the intersection of time-varying and nonlinear system dynamics, which this work aims to address.

Architectures such as Mamba [9], HGRN [22], and GLA [30] represent a new frontier in deep learning, termed selective SSMs [4, 31, 27]. Their defining characteristic, captured in our continuous-time model (1), is that their core state-space parameters (A, B, C) are not fixed but are dynamically shaped by both time-varying gating signals $\Delta(t)$ and the input sequence $x(t)$ itself. This input-selectivity is key to their performance, enabling selective context processing. Analytically, however, this places these models in a complex domain: they are neither purely Linear Time-Varying (LTV), due to the

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$x(t)$ dependence, nor are they classical nonlinear systems for which standard tools always apply without adaptation [13, 16]. While the theory for LTV system stability, including the behavior of quadratic "energy" functions V_Q (e.g., their regularity [20]), and for ISS of input-driven nonlinear systems [24] are individually well-understood, their integrated application to selective SSMs is not explored. Specifically, how does a unified "energy management" or passivity perspective constrain these input-dependent dynamics, especially when $\Delta(t)$ introduces abrupt changes? This paper addresses this crucial question by developing a rigorous framework for analyzing stability and structural properties in these advanced models. We bridge this gap by systematically applying and extending concepts from passivity theory and ISS to continuous-time selective SSMs. We investigate the conditions under which these models exhibit stable behavior and how a dissipativity assumption constrains their internal structure and gating mechanisms. Specifically, we demonstrate that even with input-dependent and potentially discontinuous gating, foundational properties of energy-like functions persist and impose significant structural constraints. Our contributions are as follows:

1. We demonstrate that the assumption of intrinsic energy dissipation is key to explaining and guaranteeing exponential forgetting of past states (Theorem 3.1), thus linking a core system property to observed memory decay phenomena. Further, we prove that the unforced system (input $= 0$) possesses an underlying minimal quadratic energy function whose defining matrix has robust AUC_{loc} regularity, accommodating discontinuous gating (Theorem 3.3). This reveals a fundamental, well-behaved energy structure irrespective of input-driven variations.
2. If a single quadratic energy function ensures passivity across all input-gating scenarios, its defining matrix must also be AUC_{loc} with non-increasing energy capacity (rank). This leads to mathematical inequalities (LMIs) and a key kernel condition: "energy-less" state directions must be output-unobservable under any gating (Theorem 4.2).
3. Continuing with the universal quadratic energy assumption, we formalize "irreversible forgetting": once a state direction is "energy-less", it remains so structurally. The system's dynamics under any gating must maintain energy consistency within this forgotten subspace, constraining how gating can influence these modes without violating passivity (Theorem 4.3).
4. We establish strong conditions for global Input-to-State Stability (ISS). If a single quadratic Lyapunov function demonstrates a uniformly negative rate of change across all input-induced system behaviors, the selective SSM is robustly stable (Theorem 5.1), which links local dissipative behavior to global system properties. This extends the analysis beyond zero-input stability (Theorem 3.1) to account for persistent inputs.

Our findings offer new theoretical insights into the regularity requirements, structural constraints imposed by passivity, and robust stability of deep selective SSMs, contributing to a more principled understanding and design of these powerful models, particularly in contexts where parameters may vary discontinuously over time, potentially frequently and abruptly, due to gating. The paper is structured as follows: Section A.1 reviews relevant prior literature, providing context for our approach and contributions. Section 2 introduces our system model and the core concepts of passivity. Section 3 studies the baseline stability guarantees and the inherent quadratic energy structure of the unforced system dynamics, highlighting the AUC_{loc} regularity of its storage matrix even with discontinuous gating. Building on this, Section 4 explores the profound implications of assuming a universal quadratic storage function, deriving parametric LMI conditions and crucial kernel constraints that restrict gating mechanisms and formalize the notion of "irreversible forgetting". Section 5 establishes sufficient conditions for global ISS, linking uniform local dissipativity across all input-selected modes to overall system robustness. Finally, Section 6 discusses the broader implications of our results for model design, connects them to practical observations, outlines limitations, and suggests avenues for future research. For the Related Work section and proofs, please check Appendix A. At the end of each proof in the Appendix, there is a *Summary* explaining its importance.

2 Preliminaries: Selective SSMs and Passivity Framework

This section introduces the continuous-time selective State-Space Models (SSMs) that are the focus of our analysis, along with the fundamental concepts from passivity and dissipativity theory.

2.1 System Definition and Notation

We consider dynamical systems evolving in continuous time $t \in \mathbb{T}$, where $\mathbb{T} \subseteq [0, +\infty)$ is a time interval. The internal state of the system at time t is denoted by $h(t) \in \mathbb{C}^N$ (or $\mathbb{R}^N$ in real-valued cases). The system interacts with its environment via an external input signal $x(t) \in \mathbb{C}^{d_{\text{in}}}$ and produces an output $y(t) \in \mathbb{C}^{d_{\text{out}}}$. For matrices and vectors, M^H denotes the Hermitian (conjugate) transpose. The standard Euclidean norm of a vector v is $\|v\|$, and the inner product of two vectors u, v is $\langle u, v \rangle = v^H u$. We denote the space of $n \times n$ Hermitian positive semidefinite matrices by $\mathbf{S}_+^n(\mathbb{C})$; for $Q \in \mathbf{S}_+^n(\mathbb{C})$, we write $Q \succeq 0$. Our work focuses on Continuous-Time Selective SSMs, inspired by architectures like Mamba [9], where the system parameters can be dynamically modulated by both an auxiliary gating signal $\Delta(t)$ and the external input $x(t)$. Formally, the dynamics are given by:

$$\begin{cases} \dot{h}(t) \;=\; A\Big(\Delta(t),\, x(t)\Big)\; h(t) \;+\; B\Big(\Delta(t),\, x(t)\Big)\; x(t), \\ y(t) \;=\; C\Big(\Delta(t),\, x(t)\Big)\; h(t). \end{cases} \tag{1}$$

Here, for each time t and input $x(t)$, $A(\Delta(t), x(t)) \in \mathbb{C}^{N \times N}$ is the state transition matrix, $B(\Delta(t), x(t)) \in \mathbb{C}^{N \times d_{\text{in}}}$ is the input matrix, and $C(\Delta(t), x(t)) \in \mathbb{C}^{d_{\text{out}} \times N}$ is the output matrix. The crucial characteristic is the explicit dependence of A, B, C on $x(t)$, which renders the system (1) generally nonlinear (specifically, input-dependent) rather than purely LTV. The signal $\Delta(t)$ allows for additional time-based variations, potentially introducing discontinuities if $\Delta(t)$ itself is discontinuous - its internal "rules" (the matrices A, B, C) can suddenly change due to the gating.

2.2 Well-Posedness and Function Space Assumptions

To ensure that the system (1) is well-defined, we assume mild regularity conditions on its coefficient matrices. For any admissible input trajectory $x(\cdot)$, let $A_{\text{eff}}(t) \coloneqq A(\Delta(t), x(t))$, $B_{\text{eff}}(t) \coloneqq B(\Delta(t), x(t))$, and $C_{\text{eff}}(t) \coloneqq C(\Delta(t), x(t))$. We assume: $A_{\text{eff}}(\cdot) \in L^1_{\text{loc}}(\mathbb{T}, \mathbb{C}^{N \times N})$ (locally integrable), $B_{\text{eff}}(\cdot) \in L^2_{\text{loc}}(\mathbb{T}, \mathbb{C}^{N \times d_{\text{in}}})$ (locally square-integrable), $C_{\text{eff}}(\cdot) \in L^2_{\text{loc}}(\mathbb{T}, \mathbb{C}^{d_{\text{out}} \times N})$ (locally square-integrable), and the input $x(\cdot) \in L^2_{\text{loc}}(\mathbb{T}, \mathbb{C}^{d_{\text{in}}})$. These conditions, particularly for A_{eff}, ensure that for any initial condition $h(t_0) = h_0$, the ODE for $h(t)$ satisfies Carathéodory conditions, guaranteeing the local existence and uniqueness of an absolutely continuous solution $h(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{T}, \mathbb{C}^N)$ [8, 5]. This framework accommodates discontinuous changes in system parameters induced by $\Delta(t)$ or $x(t)$, provided the resulting matrices remain locally integrable.

2.3 Passivity and Dissipativity

We analyze the system (1) using the concepts of passivity and dissipativity [29, 28].

Definition 2.1 (Storage Function)**.** A function $V : \mathbb{T} \times \mathbb{C}^N \to \mathbb{R}$ is a *storage function* if it is non-negative, i.e., $V(t, h) \geq 0$ for all (t, h), and typically $V(t, 0) = 0$.

Definition 2.2 (Passivity)**.** The system (1) is *passive* if there exists a storage function V such that for all $t_0, T \in \mathbb{T}$ with $t_0 \leq T$, and all admissible trajectories:

$$V\big(T, h(T)\big) - V\big(t_0, h(t_0)\big) \leq \int_{t_0}^{T} \operatorname{Re} \langle x(\tau), y(\tau) \rangle \, d\tau. \tag{2}$$

The term $\operatorname{Re} \langle x(\tau), y(\tau) \rangle$ is the supply rate.

Definition 2.3 (Strict Dissipativity (β-Strict Passivity))**.** The system (1) is *strictly dissipative* with rate $\beta > 0$ (or β-strictly passive) if there exists a storage function V such that for all $t_0, T \in \mathbb{T}$ with $t_0 \leq T$:

$$V\big(T, h(T)\big) - V\big(t_0, h(t_0)\big) \leq \int_{t_0}^{T} \operatorname{Re} \langle x(\tau), y(\tau) \rangle \, d\tau - \beta \int_{t_0}^{T} \|h(\tau)\|^2 \, d\tau. \tag{3}$$

If $\beta = 0$, this reduces to standard passivity. A positive β implies an intrinsic rate of energy dissipation related to the state norm, which is crucial for proving stronger stability properties like exponential decay. Under sufficient regularity conditions for $V(t, h(t))$, these integral inequalities have corresponding differential forms: $\frac{d}{dt} V(t, h(t)) \leq \operatorname{Re} \langle x(t), y(t) \rangle$ for passivity, and $\frac{d}{dt} V(t, h(t)) \leq \operatorname{Re} \langle x(t), y(t) \rangle - \beta \|h(t)\|^2$ for strict dissipativity.

2.4 Quadratic Storage Functions and AUC Regularity

A significant portion of our analysis focuses on *quadratic storage functions* of the form: $V_Q(t,h) = \frac{1}{2}h^H Q(t)h$, where $Q : \mathbb{T} \to \mathbf{S}_+^N(\mathbb{C})$ is a time-varying Hermitian positive semidefinite matrix. The regularity of $Q(t)$ is critical, especially when system parameters $A(t,x), B(t,x), C(t,x)$ can be discontinuous due to gating. Following the work of Morandin and Hinsen [20] on LTV systems, we consider $Q(t)$ to belong to the class of Locally Absolutely Upper Semicontinuous matrix functions, denoted $Q \in \mathrm{AUC}_{\mathrm{loc}}(\mathbb{T}, \mathbf{S}_+^N(\mathbb{C}))$. A function $Q \in \mathrm{AUC}_{\mathrm{loc}}$ is characterized by being of locally bounded variation ($\mathrm{BV}_{\mathrm{loc}}$), its derivative $\dot{Q}(t)$ exists almost everywhere and is in L^1_{loc}, and it satisfies $Q(s) - Q(r) \leq \int_r^s \dot{Q}(\tau)d\tau$ for $r \leq s$. Crucially, the singular part of its Jordan decomposition is weakly monotonically decreasing in the Loewner order. This regularity allows for $Q(t)$ to have jump discontinuities but constrains these jumps (e.g., $\lim_{\tau\to t^-} Q(\tau) \succeq Q(t) \succeq \lim_{\tau\to t^+} Q(\tau)$), making it suitable for analyzing systems with switching behavior while retaining essential structural properties like non-increasing rank over time.

3 Fundamental Stability and Structural Properties from Passivity

This section establishes baseline stability guarantees and explores the core structural properties inherent in passive selective SSMs. We begin by showing how strict dissipativity leads to exponential decay in the absence of input, and then demonstrate that the unforced dynamics possess a fundamental, well-behaved quadratic energy structure, even under discontinuous gating, by leveraging established results for passive Linear Time-Varying (LTV) systems.

3.1 Exponential Decay from Strict Dissipativity

The most fundamental stability guarantee arises when the system intrinsically dissipates energy faster than it stores it, even with zero input. This leads to exponential convergence of the state to the origin.

Theorem 3.1 (Exponential Decay from Strict Dissipativity)**.** Consider the continuous-time selective state-space model defined in Eq. (1). Suppose there exists a storage functional $V : [0,\infty) \times \mathbb{C}^N \to \mathbb{R}_{\geq 0}$ satisfying:

(i) **Strict Dissipativity Inequality:** For some constant $\beta > 0$, every admissible trajectory $\{h(\tau), x(\tau), y(\tau)\}$ on any interval $[t_0, T]$ satisfies:

$$V\big(T,\, h(T)\big) - V\big(t_0,\, h(t_0)\big) \leq \int_{t_0}^{T} \mathrm{Re}\,\langle x(\tau), y(\tau)\rangle\; d\tau - \beta \int_{t_0}^{T} \|h(\tau)\|^2\, d\tau. \tag{4}$$

(ii) **Regularity and Quadratic Bounds:** $V(t,h)$ is locally Lipschitz in h, absolutely continuous in t along system trajectories, and there exist constants $k_2 \geq k_1 > 0$ such that for all $t \geq 0$ and $h \in \mathbb{C}^N$:

$$k_1\|h\|^2 \leq V(t,h) \leq k_2\|h\|^2. \tag{5}$$

Then, for the unforced system (i.e., when $x(t) \equiv 0$ for $t \geq 0$), the state exhibits exponential decay: there exist constants $C \geq 1$ and $\gamma > 0$ such that for any initial state $h(0)$, the solution satisfies:

$$\|h(t)\| \leq C\, e^{-\gamma t}\, \|h(0)\| \quad \text{for all } t \geq 0. \tag{6}$$

Proof of Theorem 3.1. Check Appendix A.2. □

3.2 Inherent Quadratic Structure and Regularity in Unforced Dynamics

While the full selective SSM dynamics are complex due to input dependence, we now show that if the overall system possesses a passive structure (even with a general storage function), its underlying unforced dynamics ($x = 0$) necessarily inherit a well-defined and mathematically regular *quadratic* energy structure. This structure persists even when the gating signal $\Delta(t)$ introduces discontinuities. We first confirm a property of the minimal storage function definition.

Lemma 3.2 (Strict Passivity of the Minimal Available Storage Function)**.** Let $V_{\min}(t,h)$ be the minimal available storage function defined as:

$$V_{\min}(t,h) \coloneqq \sup_{\substack{S\geq t \\ \text{admissible inputs } \hat{x}(\cdot) \text{ on } [t,S]}} \left[-\int_t^S \operatorname{Re}\langle \hat{x}(\tau), \hat{y}(\tau)\rangle \, d\tau \; + \; \beta \int_t^S \|\hat{h}(\tau)\|^2 \, d\tau \right], \quad (7)$$

where $\hat{h}(\cdot)$ is the state trajectory starting at $\hat{h}(t) = h$ driven by input $\hat{x}(\cdot)$, $\hat{y}(\cdot)$ is the corresponding output, and $\beta \geq 0$ is the dissipation rate from Eq. (4). Assume the system is such that this supremum is finite for all (t,h) (which is guaranteed if there exists at least one storage function V satisfying Eq. (4)). Then $V_{\min}$ itself satisfies the strict dissipativity inequality; that is, for every admissible state-input-output trajectory $\{h(\tau), x(\tau), y(\tau)\}$ on an interval $[t_0, T]$,

$$V_{\min}\big(T, h(T)\big) \; - \; V_{\min}\big(t_0, h(t_0)\big) \;\; \leq \;\; \int_{t_0}^T \operatorname{Re}\langle x(\tau), y(\tau)\rangle \, d\tau \; - \; \beta \int_{t_0}^T \|h(\tau)\|^2 \, d\tau. \quad (8)$$

Proof of Lemma 3.2. Check Appendix A.3. □

The following theorem leverages this structure, linking the passivity of the overall selective system to the properties of its unforced counterpart using the framework of [20].

Theorem 3.3 (Existence and Regularity of Quadratic Storage for Unforced Dynamics Amidst Gating Switches)**.** Consider the continuous-time selective state-space model (1) defined on a time interval $\mathbb{T}$. Assume the coefficient matrices satisfy mild regularity conditions ensuring well-posedness: (i) $A(\Delta(\cdot), x(\cdot)) \in L^1_{\text{loc}}(\mathbb{T}, \mathbb{C}^{N\times N})$ (ii) $B(\Delta(\cdot), x(\cdot)) \in L^2_{\text{loc}}(\mathbb{T}, \mathbb{C}^{N\times d_{\text{in}}})$ (iii) $C(\Delta(\cdot), x(\cdot)) \in L^2_{\text{loc}}(\mathbb{T}, \mathbb{C}^{d_{\text{out}}\times N})$ for admissible inputs $x(\cdot)$, allowing $\Delta(t)$ to induce discontinuities (e.g., piecewise constant changes) in A, B, C provided the resulting matrix functions remain in the specified local Lebesgue spaces. Suppose there exists any storage functional $V : \mathbb{T} \times \mathbb{C}^N \to \mathbb{R}_{\geq 0}$ ($V \geq 0, V(t,0) = 0$) satisfying the (potentially strict, $\beta \geq 0$) passivity inequality for the full selective system (1):

$$V\big(T,\, h(T)\big) - V\big(t_0,\, h(t_0)\big) \leq \int_{t_0}^T \operatorname{Re}\langle x(\tau), y(\tau)\rangle \, d\tau - \beta \int_{t_0}^T \|h(\tau)\|^2 \, d\tau \quad (9)$$

for all admissible state-input-output trajectories $\{h(\tau), x(\tau), y(\tau)\}$ and all $t_0 \leq T$ in $\mathbb{T}$. Then, the following conclusions hold regarding the structure induced by these assumptions, even in the presence of gating switches:

(a) **Passivity of Unforced LTV System:** The unforced linear time-varying (LTV) system, defined by isolating the dynamics when $x(t) \equiv 0$:

$$\begin{cases} \dot{h}(t) \;=\; A_0(t)\, h(t) & \text{where } A_0(t) = A(\Delta(t), 0) \in L^1_{\text{loc}}(\mathbb{T}, \mathbb{C}^{N\times N}) \\ y_0(t) \;=\; C_0(t)\, h(t) & \text{where } C_0(t) = C(\Delta(t), 0) \in L^2_{\text{loc}}(\mathbb{T}, \mathbb{C}^{d_{\text{out}}\times N}) \end{cases} \quad (10)$$

is passive in the sense of [20, Def 1.1]. The function V serves as a valid, although potentially non-quadratic, storage function for this LTV system.

(b) **Existence of Minimal Quadratic Storage for Unforced LTV:** Consequently, by the theory for passive LTV systems [20], the unforced LTV system (10) admits a minimal available storage function $V_{a,0}(t,h)$ which is necessarily quadratic in the state h:

$$V_{a,0}(t,h) = \tfrac{1}{2} h^H Q_0(t) h \quad (11)$$

for a unique matrix function $Q_0 : \mathbb{T} \to \mathbf{S}^N_+(\mathbb{C})$. This holds despite potential discontinuities in $A_0(t)$ and $C_0(t)$ induced by $\Delta(t)$.

(c) **AUC Regularity of $Q_0(t)$:** Furthermore, the matrix function $Q_0(t)$ associated with the minimal quadratic storage $V_{a,0}$ possesses AUC_{loc} regularity, i.e., $Q_0 \in \text{AUC}_{\text{loc}}(\mathbb{T}, \mathbf{S}^N_+(\mathbb{C}))$. This specific regularity class robustly handles potential jump discontinuities from $\Delta(t)$ while ensuring essential structural properties like locally bounded variation and weakly decreasing singular part.

Finally, if the original storage function V for the full selective system satisfies the stricter conditions of Theorem 3.1 (i.e., strict passivity $\beta > 0$ and quadratic bounds (5)), then the state $h(t)$ of the unforced system (10) is guaranteed to decay exponentially, as proven in Theorem 3.1.

Proof of Theorem 3.3. Check Appendix A.4. □

4 Constraints Imposed by Universal Quadratic Passivity on Gating Mechanisms

While Section 3 established properties based on the existence of any storage function, particularly focusing on the unforced dynamics, this section investigates the considerably stronger assumption that a *single*, time-varying quadratic storage function $V_Q(t, h) = \frac{1}{2}h^H Q(t)h$ guarantees passivity for the *entire* selective state-space model across all admissible inputs $x(t)$ and the corresponding gating signals $\Delta(t)$. We explore the profound implications this assumption has for the regularity of the storage matrix $Q(t)$ itself and the constraints it imposes on the permissible system dynamics $(A(t, x), B(t, x), C(t, x))$ generated by the input-dependent gating mechanism.

4.1 Regularity and Rank Monotonicity of Universal Quadratic Storage

If a single quadratic function V_Q is capable of certifying passivity regardless of the input-driven variations in system parameters, the matrix $Q(t)$ defining this function must possess inherent structural regularity and obey specific monotonicity properties.

Theorem 4.1 (Regularity and Rank of Universal Quadratic Storage)**.** Consider the continuous-time selective state-space model (1) under the standard regularity assumptions. Suppose there exists a time-varying quadratic storage function $V_Q(t, h) = \frac{1}{2}h^H Q(t)h$ (with $Q : \mathbb{T} \to \mathbf{S}_+^N(\mathbb{C})$) that satisfies the passivity inequality:

$$V_Q\big(T,\, h(T)\big) - V_Q\big(t_0,\, h(t_0)\big) \leq \int_{t_0}^{T} \operatorname{Re}\langle x(\tau), y(\tau)\rangle \, d\tau - \beta \int_{t_0}^{T} \|h(\tau)\|^2 \, d\tau \tag{12}$$

for some $\beta \geq 0$, valid for *all* admissible state-input-output trajectories $\{h(\tau), x(\tau), y(\tau)\}$ generated by any admissible input $x(\cdot)$. Then, the matrix function $Q(t)$ must satisfy the following properties:

(a) **AUC Regularity:** $Q(t)$ must belong to the class of locally absolutely upper semicontinuous matrix functions, i.e., $Q \in \mathrm{AUC}_{\mathrm{loc}}(\mathbb{T}, \mathbf{S}_+^N(\mathbb{C}))$.

(b) **Rank Monotonicity:** The rank of $Q(t)$, denoted $r(t) = \operatorname{rank}(Q(t))$, must be weakly monotonically non-increasing over time $t \in \mathbb{T}$.

Proof of Theorem 4.1. Check Appendix A.5. □

4.2 Parametric LMI and Kernel Constraints on Gating

The existence of a universal quadratic storage function V_Q translates into concrete algebraic constraints that the system matrices $A(t, x), B(t, x), C(t, x)$ must satisfy *parametrically* for all possible input values x. These constraints take the form of a Linear Matrix Inequality (LMI) and lead to important conditions on how the output matrix $C(t, x)$ interacts with the kernel of the storage matrix $Q(t)$.

Theorem 4.2 (Passivity Constraints on Gating via Kernel Conditions)**.** Consider the continuous-time selective state-space model (1) under the standard regularity assumptions. Assume the system's passivity is guaranteed by a *single*, time-varying quadratic storage function $V_Q(t, h) = \frac{1}{2}h^H Q(t)h$ (with $Q : \mathbb{T} \to \mathbf{S}_+^N(\mathbb{C})$ satisfying $Q \in \mathrm{AUC}_{\mathrm{loc}}$ and non-increasing rank, per Theorem 4.1) that fulfills the passivity inequality (12) for some $\beta \geq 0$ and for *all* admissible inputs $x(\cdot)$ and trajectories. Then, the following necessary conditions hold:

(a) **Parametric LMI Condition:** For almost every $t \in \mathbb{T}$, the LMI associated with passivity must hold *parametrically* for all admissible input values $x \in \mathbb{C}^{d_{\text{in}}}$ that can occur at time t:

$$\mathcal{L}(t,x) \coloneqq \begin{bmatrix} \dot{Q}(t) + Q(t)A(t,x) + A(t,x)^H Q(t) + 2\beta I & Q(t)B(t,x) - C(t,x)^H \\ B(t,x)^H Q(t) - C(t,x) & 0 \end{bmatrix} \preceq 0 \tag{13}$$

where $A(t,x) = A(\Delta(t),x)$, $B(t,x) = B(\Delta(t),x)$, $C(t,x) = C(\Delta(t),x)$, and $\dot{Q}(t)$ exists a.e. [1]

(b) **Universal Kernel Condition for C:** The output matrix $C(t,x)$ must map the kernel of $Q(t)$ to zero, *uniformly* for all admissible input values x:

$$\forall v \in \operatorname{Ker}\big(Q(t)\big), \quad C\big(\Delta(t),x\big)v = 0 \quad \text{for all admissible } x \text{ at time } t, \text{ a.e. } t \in \mathbb{T}. \tag{14}$$

(c) **Implicit Constraints on A and B:** The $(1,1)$ block inequality $\dot{Q} + QA + A^H Q + 2\beta I \preceq 0$ and the off-diagonal block constraint (related to $QB - C^H$) must also hold parametrically for all admissible x, implicitly restricting the choices of $A(t,x)$ and $B(t,x)$ based on $Q(t)$ and the constrained $C(t,x)$.

Proof of Theorem 4.2. Check Appendix A.6. □

4.3 Irreversible Forgetting and Energy Consistency in Kernel Subspace

The constraints derived above, particularly the non-increasing rank of $Q(t)$ and the universal kernel condition for $C(t,x)$, suggest a form of structural irreversibility associated with the modes that fall into the kernel of the storage matrix $Q(t)$. We now explore this concept of "irreversible forgetting" from the perspective of the universal storage function V_Q and examine the energy balance required within this forgotten subspace.

Theorem 4.3 (Inertness of Forgotten Modes and Gating Constraints from Passivity)**.** Consider the continuous-time selective state-space model (1) under standard regularity assumptions. Assume its passivity is guaranteed by a *single*, time-varying quadratic storage function $V_Q(t,h) = \frac{1}{2}h^H Q(t)h$ (with $Q \in \mathrm{AUC}_{\mathrm{loc}}(\mathbb{T}, \mathbf{S}_+^N(\mathbb{C}))$ having non-increasing rank) satisfying inequality (12) for some $\beta \geq 0$ and for *all* admissible inputs $x(\cdot)$. Then, the following properties regarding "forgotten modes" (states in $\operatorname{Ker}(Q(t))$) hold:

1. **Kernel Non-Shrinking (Irreversible Forgetting):** The subspace $\operatorname{Ker}(Q(t))$ is non-decreasing over time: for any $t_1 \leq t_2$ in $\mathbb{T}$, $\operatorname{Ker}(Q(t_1)) \subseteq \operatorname{Ker}(Q(t_2))$. A state direction v, once in the kernel at t_1, remains in the kernel subspace for all $t_2 \geq t_1$, from the perspective of $Q(t)$.

2. **Energy Consistency within the Kernel:** For any state $h(t) \in \operatorname{Ker}(Q(t))$ at a time $t \in \mathbb{T}$ where $\dot{Q}(t)$ exists, the energy balance condition holds:

$$h(t)^H \dot{Q}(t) h(t) + 2\beta \|h(t)\|^2 \leq 0. \tag{15}$$

This ensures the evolution $\dot{Q}(t)$ within the kernel is consistent with the required dissipation β. If $\beta = 0$, $h^H \dot{Q} h \leq 0$; if $\beta > 0$, $h^H \dot{Q} h \leq -2\beta\|h\|^2 < 0$ (for $h \neq 0$).

3. **Constraint on Dynamics Violating Kernel Structure:** Any input $x^*(t)$ inducing dynamics $A^*(t), B^*(t)$ that would require a state $h(t) \in \operatorname{Ker}(Q(t))$ to evolve in a way inconsistent with the LMI (13) (e.g., appearing to "gain energy" according to V_Q) is inadmissible under the assumption of universal passivity with V_Q. The dynamics must respect the energy structure defined by $Q(t)$ and $\dot{Q}(t)$ within the kernel.

Proof of Theorem 4.3. Check Appendix A.7. □

[1] The $2\beta I$ term arises from incorporating the strict passivity term $-\beta\|h\|^2$ into the LMI, assuming the state h corresponds to the first block.

5 Robust Stability under Input-Driven Dynamics

Sections 3 and 4 primarily focused on the implications of passivity, establishing baseline stability for the unforced system and exploring the constraints imposed by assuming a universal quadratic storage function. We now shift our focus to the behavior of the selective state-space model (1) under the influence of non-zero external inputs $x(t)$. The goal is to establish conditions that guarantee robust stability, ensuring that the system state remains bounded for bounded inputs and converges to a region determined by the input magnitude. This leads us to the concept of ISS.

5.1 Global Input-to-State Stability from Uniform Dissipativity

To guarantee robust stability for the selective SSM in the presence of arbitrary bounded inputs, we need conditions stronger than basic passivity. Specifically, we seek conditions ensuring that the system's internal dynamics are sufficiently contractive, uniformly across all possible configurations induced by the input $x(t)$, to overcome the driving effect of the input. This subsection presents such a condition based on the existence of a common quadratic Lyapunov function demonstrating uniform dissipativity.

Theorem 5.1 (Global Stability from Uniform Local Dissipativity)**.** Consider the continuous-time selective state-space model (1) under the standard regularity assumptions. Assume further that:

(i) There exists a time-varying quadratic Lyapunov function candidate $V_Q(t,h) = \frac{1}{2}h^H Q(t)h$, where $Q : \mathbb{T} \to \mathbf{S}_+^N(\mathbb{C})$ satisfies:

- $Q \in W_{\text{loc}}^{1,1}(\mathbb{T}, \mathbf{S}_+^N(\mathbb{C}))$ (absolutely continuous locally).
- $Q(t)$ is uniformly positive definite and bounded: there exist constants $k_2 \geq k_1 > 0$ such that $k_1 I \preceq Q(t) \preceq k_2 I$ for all $t \in \mathbb{T}$.

(ii) The dynamics satisfy a uniform dissipativity condition with respect to V_Q: There exists a constant $\delta > 0$ such that for almost every $t \in \mathbb{T}$ and for **all** admissible input values $x \in \mathbb{C}^{d_{\text{in}}}$ that can occur at time t, the following matrix inequality holds:

$$\dot{Q}(t) + Q(t)A\big(\Delta(t), x\big) + A\big(\Delta(t), x\big)^H Q(t) \preceq -2\delta Q(t) \tag{16}$$

(This implies that for any fixed x, the homogeneous system $\dot{h} = A(\Delta(t), x)h$ is uniformly exponentially stable with V_Q as a Lyapunov function decaying at rate δ).

(iii) The input matrix $B(\Delta(t), x)$ is uniformly bounded: There exists a constant $M_B > 0$ such that $\|B(\Delta(t), x)\|_2 \leq M_B$ for all t and admissible x.

Then, the selective state-space system (1) is globally ISS with respect to the input $x(t)$. Specifically, there exist constants $\tilde{C} \geq 1$, $\tilde{\gamma} > 0$, and a class $\mathcal{K}$ gain function σ such that for any initial state $h(t_0)$ and any admissible input $x(\cdot)$, the solution satisfies:

$$\|h(t)\| \leq \tilde{C}e^{-\tilde{\gamma}(t-t_0)}\|h(t_0)\| + \sigma\left(\sup_{t_0 \leq \tau \leq t} \|x(\tau)\|\right) \quad \text{for all } t \geq t_0. \tag{17}$$

Proof of Theorem 5.1. Check Appendix A.8. □

6 Discussion

In this paper, we established a bridge between the rigorous analytical tools of control theory and the complex, input-dependent dynamics of modern selective SSMs. By employing passivity theory and Input-to-State Stability (ISS), we explained the conditions under which these models, even when subject to discontinuous gating signals, can exhibit stable and predictable behavior. Our aim has been not only to characterize these systems but also to lay the groundwork for more principled design and interpretation of these models. The theoretical journey of this work began by establishing that even a highly adaptive, selective SSM, if it possesses an overall passive structure (certified by any valid storage function), inherently contains a well-behaved core. Specifically, its unforced dynamics (the system's behavior in the absence of external input) are governed by an underlying minimal quadratic energy function whose defining matrix, $Q_0(t)$, exhibits AUC_{loc} regularity (Theorem 3.3). This is a crucial foundational insight: it suggests that, irrespective of the complexities introduced

by input-driven selection, there is an intrinsic, mathematically regular energy landscape associated with the system's "default" state evolution. This "default" evolution describes how the system's state changes over time when driven solely by its initial condition and internal parameters, without any active external input. Consequently, the way these models are initialized gains critical significance, as the initial parameters are responsible for defining this baseline energy behavior. Furthermore, if this underlying system is strictly dissipative, it naturally exhibits exponential forgetting of its initial state (Theorem 3.1). Building upon this, we explored the more strict scenario where a single, universal quadratic storage function $V_Q(t,h) = \frac{1}{2}h^H Q(t)h$ ensures passivity across all possible input-gating configurations. This stronger assumption opens a cascade of structural implications for both the storage matrix $Q(t)$ itself and the permissible gating mechanisms. $Q(t)$ must then also possess $\mathrm{AUC}_{\mathrm{loc}}$ regularity and, critically, its rank must be non-increasing over time (Theorem 4.1). This rank monotonicity leads to the concept of an "energy-less" subspace, $\mathrm{Ker}(Q(t))$, which can only grow or remain constant, signifying a structural irreversibility in how the system stores energy according to V_Q. The consequences for the gating logic are profound: the system matrices $A(t,x)$, $B(t,x)$, and $C(t,x)$ must parametrically satisfy a Linear Matrix Inequality for all input values x. A striking outcome is that the output matrix $C(t,x)$ is constrained to map any state in $\mathrm{Ker}(Q(t))$ to zero, regardless of the input x (Theorem 4.2). This means that directions deemed "forgotten" or "energy-less" by the universal storage function V_Q cannot be made observable through input-driven gating if this unified passive structure is to be maintained. Moreover, the evolution of $Q(t)$ and the system dynamics within this kernel subspace must adhere to strict energy consistency rules, formalizing an "irreversible forgetting" from the perspective of V_Q (Theorem 4.3). Finally, this line of reasoning culminates in providing conditions for global ISS: if a common quadratic Lyapunov function V_Q can certify a uniform rate of energy decrease across all input-selected system modes, then the entire selective SSM is robustly stable to bounded inputs (Theorem 5.1). This beautifully links the uniform stability of all "local" operational modes (selected by x) to the global robustness of the overall gated system.

These theoretical constructs, while abstract, offer a principled lens through which to interpret and guide practical aspects of SSM development. The significance of careful initialization strategies, such as HiPPO [10] in S4/Mamba models [11, 9], finds a theoretical anchor in the properties of $Q_0(t)$ for the unforced system. A "good" initialization establishes a favorable baseline energy landscape that promotes stable learning and operation. The observed phenomena of memory decay and the ability of SSMs to handle long contexts can be related to the guarantees of exponential decay and ISS, which ensure that the influence of past states and distant inputs diminishes in a controlled manner. The notion of irreversible forgetting tied to a non-shrinking $\mathrm{Ker}(Q(t))$ and the associated energy consistency constraints offers a novel, if speculative, control-theoretic perspective on catastrophic forgetting in continual learning. If adapting to new, conflicting information requires the system to violate the structural constraints necessary for maintaining a single, universal V_Q (e.g., by needing to make a "forgotten" mode in $\mathrm{Ker}(Q(t))$ observable again, or by needing to re-allocate energy in a way that disrupts the established balance), the pre-existing passive energy structure might be broken, leading to the loss of previously learned information. It is important, also, to acknowledge the limitations of the current work. Our analysis is rooted in continuous-time dynamics. While continuous-time models often capture the essence of system behavior and provide foundational principles, translating these directly to discrete-time implementations, which are standard in deep learning, requires careful consideration of sampling effects and discrete-time stability theory.

Future research offers several promising directions. A primary focus is developing constructive methodologies, moving from analysis to synthesis by using these insights to guide architectural design or create regularization techniques that promote the derived LMI conditions or uniform dissipativity for ISS. Investigating input-dependent storage matrices $Q(t,x)$ is vital for modeling adaptive energy management. Essential for practical SSMs is the formulation of discrete-time analogues of our theorems. Designing experiments to observe $Q(t)$ or kernel dynamics during training would also offer crucial validation. Longer-term research includes non-quadratic Lyapunov functions and the analysis of interconnected selective SSMs. In conclusion, this work has aimed to demonstrate the utility of a rigorous control-theoretic perspective for dissecting the behavior of continuous-time selective SSMs. By providing a formal language to discuss their stability, regularity, and the constraints on their adaptive mechanisms, we hope to contribute to a deeper understanding and the continued development of robust, reliable, and interpretable AI systems for sequential data processing.

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A Technical Appendices and Supplementary Material

A.1 Related Work

The analysis of stability and energy-based properties in dynamical systems has a rich history, providing crucial tools for understanding systems ranging from classical mechanics to modern AI. Our work on continuous-time selective State-Space Models (SSMs) builds upon several key research streams.

A.1.1 Dissipativity, Passivity, and LTV Systems

The foundational theory of dissipative systems, pioneered by Willems [29], provides a general framework for analyzing systems based on energy-like storage functions and supply rates. Passivity, a special case where the supply rate is the input-output inner product, is central to understanding robust stability and interconnection [28]. The Kalman-Yakubovich-Popov (KYP) lemma established a vital link between frequency-domain passivity conditions and state-space properties for LTI systems [15, 21], which has been extended to LTV systems, often involving time-varying Riccati equations or differential/integral inequalities [1]. Crucially for our work, recent research by Morandin and Hinsen [20] has rigorously investigated quadratic storage functions for passive LTV systems under minimal regularity assumptions. Our work leverages these findings by demonstrating that the unforced dynamics of our selective SSMs can be treated as such an LTV system, thereby inheriting these structural properties for an underlying quadratic energy form, even with discontinuous gating.

A.1.2 Stability of Time-Varying and Switched Systems

Analyzing LTV systems inherently involves understanding the interplay between instantaneous ("frozen-time") stability and the rate of parameter variation [7]. For systems with discontinuous parameter changes, such as switched linear systems (which share characteristics with our gating mechanism $\Delta(t)$), tools like total variation analysis of the system matrix $A(t)$ [6, 18] and Lyapunov methods involving common or multiple Lyapunov functions [2] are relevant. Filippov's theory for differential equations with discontinuous right-hand sides [8] and Carathéodory conditions [5] provide the formalisms for solution existence under the weak regularity (L^p_{loc}) we assume for our system matrices $A(t,x), B(t,x), C(t,x)$. This L^p_{loc} condition essentially means that while the system matrices can be discontinuous, their p-th power remains integrable over any finite time interval, ensuring solutions are well-defined. Our work builds on this by analyzing how a single, potentially AUC_{loc} quadratic storage function can persist and constrain dynamics despite such discontinuities induced by input-dependent gating.

A.1.3 Input-to-State Stability (ISS) for Nonlinear Systems

The ISS framework, developed by Sontag [26, 25], provides a robust notion of stability for nonlinear systems subject to external inputs, characterized by ISS-Lyapunov functions. It quantifies how the system state is affected by both initial conditions and input magnitudes. While classical ISS theory often assumes smooth dynamics, its principles are highly relevant for our selective SSMs, where the input $x(t)$ makes the system effectively nonlinear. Our contribution (Theorem 5.1) adapts ISS concepts by seeking a common quadratic Lyapunov function that ensures stability uniformly across all input-induced dynamics, providing a specific condition under which these complex, input-modulated systems are globally robustly stable.

A.1.4 Stability and Dynamics of Modern SSMs in Deep Learning

Recent deep learning SSMs like S4 [11], S5 [23], and particularly Mamba [9] (and its variants), have shown remarkable performance. Theoretical analyses of these models are emerging. Halloran et al. [12] analyzed Mamba's stability via Lyapunov exponents, showing non-positive maximal exponents, implying robustness to small perturbations. Our work complements this by providing conditions for exponential decay and ISS from an energy/Lyapunov function perspective under specific passivity assumptions, offering a different angle on stability for the continuous-time selective formulation. Some other works connected selective SSMs to Controlled Differential Equations (CDEs) and Rough Path Theory to explain their expressivity [17, 19, 4]. While our focus is on stability and passivity using more classical control-theoretic tools, these works show the rich mathematical underpinnings of these models from another perspective.

A.2 Proof of Theorem 3.1

We are given the existence of a storage function $V(t,h)$ satisfying the strict dissipativity inequality (4) and the quadratic bounds (5). The regularity assumption allows us to consider the differential version of the inequality. Along any trajectory of the system (1), the time derivative of V satisfies:

$$\frac{d}{dt}V\big(t,h(t)\big) \leq \operatorname{Re}\langle x(t), y(t)\rangle - \beta\|h(t)\|^2 \tag{18}$$

almost everywhere. This follows from differentiating the integral inequality or directly from the definition of dissipativity via differential supply rates. Now, consider the unforced system, where $x(t) \equiv 0$ for all $t \geq 0$. The system dynamics become $\dot{h}(t) = A(\Delta(t),0)h(t)$, which is an LTV system. The corresponding output is $y(t) = C(\Delta(t),0)h(t)$. Therefore, the input-output power term $\operatorname{Re}\langle x(t), y(t)\rangle = \operatorname{Re}\langle 0, y(t)\rangle = 0$.

Substituting $x(t) = 0$ into the differential inequality (18), we get:

$$\frac{d}{dt}V\big(t,h(t)\big) \leq -\beta\|h(t)\|^2 \tag{19}$$

for the trajectories of the unforced system.

We can relate $\|h(t)\|^2$ back to $V(t, h(t))$ using the upper quadratic bound from (5): $\|h(t)\|^2 \geq \frac{1}{k_2} V(t, h(t))$. Substituting this into (19):

$$\frac{d}{dt} V\big(t, h(t)\big) \leq -\frac{\beta}{k_2} V\big(t, h(t)\big). \tag{20}$$

Let $\Phi(t) = V(t, h(t))$. This is a scalar non-negative function satisfying the differential inequality $\dot{\Phi}(t) \leq -\gamma' \Phi(t)$, where $\gamma' = \beta/k_2 > 0$. By the Comparison Lemma (a consequence of Grönwall's inequality), this implies:

$$\Phi(t) = V\big(t, h(t)\big) \leq V\big(0, h(0)\big)\, e^{-\gamma' t} \quad \text{for all } t \geq 0. \tag{21}$$

Now, we use both quadratic bounds from (5):

- Lower bound on $V(t, h(t))$: $k_1 \|h(t)\|^2 \leq V(t, h(t))$.
- Upper bound on $V(0, h(0))$: $V(0, h(0)) \leq k_2 \|h(0)\|^2$.

Combining these with the decay inequality (21):

$$k_1 \|h(t)\|^2 \leq V\big(t, h(t)\big) \leq V\big(0, h(0)\big)\, e^{-\gamma' t} \leq k_2 \|h(0)\|^2\, e^{-\gamma' t}. \tag{22}$$

Dividing by $k_1 > 0$:

$$\|h(t)\|^2 \leq \frac{k_2}{k_1} \|h(0)\|^2\, e^{-\gamma' t}. \tag{23}$$

Taking the square root of both sides:

$$\|h(t)\| \leq \sqrt{\frac{k_2}{k_1}}\, \|h(0)\|\, e^{-(\gamma'/2) t}. \tag{24}$$

Defining $C = \sqrt{k_2/k_1} \geq 1$ (since $k_2 \geq k_1 > 0$) and $\gamma = \gamma'/2 = \beta/(2k_2) > 0$, we obtain the desired exponential decay:

$$\|h(t)\| \leq C\, e^{-\gamma t}\, \|h(0)\|. \tag{25}$$

This completes the proof.

Summary: This theorem establishes that intrinsic energy dissipation ($\beta > 0$), when coupled with a storage function V that is quadratically comparable to the state norm, guarantees exponential stability of the unforced system ($x = 0$). This signifies a fundamental "forgetting" capability, ensuring that the influence of the initial state diminishes exponentially over time in the absence of external input, regardless of the specific (potentially non-quadratic) nature of V.

A.3 Proof of Lemma 3.2

Let $\{h(\tau), x(\tau), y(\tau)\}$ be an admissible trajectory on the interval $[t_0, T]$. Let $h_0 \coloneqq h(t_0)$ and $h_T \coloneqq h(T)$. From the definition (7), we have:

$$V_{\min}(t_0, h_0) = \sup_{\substack{S \geq t_0 \\ \tilde{x}(\cdot) \text{ on } [t_0, S]}} \left[-\int_{t_0}^{S} \mathrm{Re}\, \langle \tilde{x}(\tau), \tilde{y}(\tau) \rangle\, d\tau \;+\; \beta \int_{t_0}^{S} \|\tilde{h}(\tau)\|^2\, d\tau \right], \tag{26}$$

where $\tilde{h}(t_0) = h_0$. Consider any admissible input trajectory $\tilde{x}(\cdot)$ defined on $[t_0, S]$ with $S \geq T$. We can split the integral into two parts: $[t_0, T]$ and $[T, S]$. Let $\tilde{h}(\cdot)$ and $\tilde{y}(\cdot)$ be the state and output corresponding to $\tilde{x}(\cdot)$ starting from $\tilde{h}(t_0) = h_0$.

$$\begin{aligned} -\int_{t_0}^{S} \mathrm{Re}\, \langle \tilde{x}(\tau), \tilde{y}(\tau) \rangle\, d\tau + \beta \int_{t_0}^{S} \|\tilde{h}(\tau)\|^2\, d\tau & \\ = \left(-\int_{t_0}^{T} \mathrm{Re}\, \langle \tilde{x}(\tau), \tilde{y}(\tau) \rangle\, d\tau + \beta \int_{t_0}^{T} \|\tilde{h}(\tau)\|^2\, d\tau \right) & \\ + \left(-\int_{T}^{S} \mathrm{Re}\, \langle \tilde{x}(\tau), \tilde{y}(\tau) \rangle\, d\tau + \beta \int_{T}^{S} \|\tilde{h}(\tau)\|^2\, d\tau \right). & \end{aligned} \tag{27}$$

The second term, integrated from T to S, depends on the input $\tilde{x}(\cdot)$ on $[T, S]$ and the state $\tilde{h}(T)$ reached at time T. Now, consider specifically the set of input trajectories $\tilde{x}(\cdot)$ on $[t_0, S]$ (where $S \geq T$) that coincide with the given input $x(\cdot)$ on the interval $[t_0, T]$. For such trajectories, the state $\tilde{h}(\tau)$ will coincide with $h(\tau)$ on $[t_0, T]$ (by uniqueness of solutions), so $\tilde{h}(T) = h(T) = h_T$. Let $\hat{x}(\cdot)$ denote the portion of such $\tilde{x}(\cdot)$ on the interval $[T, S]$. The corresponding state trajectory on $[T, S]$, let's call it $\hat{h}(\cdot)$, starts at $\hat{h}(T) = h_T$. For these specific concatenated trajectories, Eq. (27) becomes:

$$\begin{aligned} -\int_{t_0}^{S} \operatorname{Re}\langle \tilde{x}(\tau), \tilde{y}(\tau)\rangle \, d\tau + \beta \int_{t_0}^{S} \|\tilde{h}(\tau)\|^2 \, d\tau \\ = \left(-\int_{t_0}^{T} \operatorname{Re}\langle x(\tau), y(\tau)\rangle \, d\tau + \beta \int_{t_0}^{T} \|h(\tau)\|^2 \, d\tau \right) \\ + \left(-\int_{T}^{S} \operatorname{Re}\langle \hat{x}(\tau), \hat{y}(\tau)\rangle \, d\tau + \beta \int_{T}^{S} \|\hat{h}(\tau)\|^2 \, d\tau \right). \end{aligned} \tag{28}$$

The first term in (28) is fixed by the given trajectory on $[t_0, T]$. Let's denote it by $C_{t_0,T}$:

$$C_{t_0,T} \coloneqq -\int_{t_0}^{T} \operatorname{Re}\langle x(\tau), y(\tau)\rangle \, d\tau + \beta \int_{t_0}^{T} \|h(\tau)\|^2 \, d\tau. \tag{29}$$

The second term is the quantity maximized in the definition of $V_{\min}(T, h_T)$, taken over the specific continuation $(\hat{x}, \hat{h}, \hat{y})$ from T to S. The supremum in the definition of $V_{\min}(t_0, h_0)$ is taken over all admissible inputs $\tilde{x}(\cdot)$ starting at t_0. This supremum must be greater than or equal to the supremum taken over the subset of inputs that match $x(\cdot)$ on $[t_0, T]$. Therefore,

$$\begin{aligned} V_{\min}(t_0, h_0) &\geq \sup_{\substack{S \geq T \\ \hat{x}(\cdot) \text{ on } [T,S]}} \left[C_{t_0,T} + \left(-\int_{T}^{S} \operatorname{Re}\langle \hat{x}(\tau), \hat{y}(\tau)\rangle \, d\tau + \beta \int_{T}^{S} \|\hat{h}(\tau)\|^2 \, d\tau \right) \right] \\ &= C_{t_0,T} + \sup_{\substack{S \geq T \\ \hat{x}(\cdot) \text{ on } [T,S]}} \left[-\int_{T}^{S} \operatorname{Re}\langle \hat{x}(\tau), \hat{y}(\tau)\rangle \, d\tau + \beta \int_{T}^{S} \|\hat{h}(\tau)\|^2 \, d\tau \right] \\ &= C_{t_0,T} + V_{\min}(T, h_T). \end{aligned}$$

Here, the supremum in the second line is exactly the definition of $V_{\min}(T, h_T)$ since $\hat{h}(T) = h_T$. Substituting back the definition of $C_{t_0,T}$:

$$V_{\min}(t_0, h_0) \geq \left(-\int_{t_0}^{T} \operatorname{Re}\langle x(\tau), y(\tau)\rangle \, d\tau + \beta \int_{t_0}^{T} \|h(\tau)\|^2 \, d\tau \right) + V_{\min}(T, h_T).$$

Rearranging this inequality gives the desired result:

$$V_{\min}(T, h_T) - V_{\min}(t_0, h_0) \leq \int_{t_0}^{T} \operatorname{Re}\langle x(\tau), y(\tau)\rangle \, d\tau - \beta \int_{t_0}^{T} \|h(\tau)\|^2 \, d\tau.$$

This holds for any admissible trajectory on $[t_0, T]$.

A.4 Proof of Theorem 3.3

(a) **Passivity of the Unforced LTV System.** Define $A_0(t) \coloneqq A(\Delta(t), 0)$ and $C_0(t) \coloneqq C(\Delta(t), 0)$. The assumed regularity implies $A_0 \in L^1_{\text{loc}}$ and $C_0 \in L^2_{\text{loc}}$. The corresponding $B_0(t) = B(\Delta(t), 0)$ is in L^2_{loc} and $D_0(t) = 0$. These conditions ensure that the unforced ODE $\dot{h}(t) = A_0(t)h(t)$ is well-posed and defines an LTV system within the framework of [20]. Let $\{h(\tau), 0, y_0(\tau)\}$ be a trajectory of the unforced system (10). This is also a trajectory of the full system (1) with input $x(\tau) \equiv 0$. Substituting $x(\tau) = 0$ into the assumed passivity inequality (9):

$$\begin{aligned} V\big(T,\, h(T)\big) - V\big(t_0,\, h(t_0)\big) &\leq \int_{t_0}^{T} \operatorname{Re}\langle 0, y_0(\tau)\rangle \, d\tau - \beta \int_{t_0}^{T} \|h(\tau)\|^2 \, d\tau \\ &= 0 - \beta \int_{t_0}^{T} \|h(\tau)\|^2 \, d\tau \leq 0 \quad (\text{since } \beta \geq 0). \end{aligned}$$

This inequality, $V(T, h(T)) - V(t_0, h(t_0)) \leq 0$, holds along all trajectories of the unforced system. Since $V \geq 0$ and $V(t, 0) = 0$, it satisfies the requirements of [20, Def 1.1] for being a storage function for the LTV system (10). Thus, the unforced system is passive.

(b) **Existence of Minimal Quadratic Storage $V_{a,0}$.** Since the unforced system (10) is a passive LTV system satisfying the L^p_{loc} regularity framework of [20], we apply their Corollary 4.3. This theorem states that the available storage function $V_{a,0}$ (defined via their Def 4.1, analogous to Lemma 3.2 but with zero supply rate relevant for the unforced system analysis) is finite and is a quadratic form in the state h. Thus, there exists a unique matrix function $Q_0 : \mathbb{T} \to \mathbf{S}^N_+(\mathbb{C})$ such that $V_{a,0}(t, h) = \frac{1}{2} h^H Q_0(t) h$. This holds regardless of discontinuities in $A_0(t)$ or $C_0(t)$ induced by $\Delta(t)$, as long as the L^p_{loc} conditions are met.

(c) **AUC Regularity of $Q_0(t)$.** Corollary 4.4 in [20] further states that the matrix $Q_0(t)$ inducing the minimal quadratic available storage $V_{a,0}$ belongs to the function class $\mathrm{AUC}_{\mathrm{loc}}(\mathbb{T}, \mathbf{S}^n_+(\mathbb{C}))$. This class ensures Q_0 is $\mathrm{BV}_{\mathrm{loc}}$, its derivative exists a.e. and is L^1_{loc}, and its jump discontinuities are constrained (weakly decreasing singular part, satisfying $\lim_{\tau \to t^-} Q_0(\tau) \succeq Q_0(t) \succeq \lim_{\tau \to t^+} Q_0(\tau)$). This $\mathrm{AUC}_{\mathrm{loc}}$ property guarantees a well-behaved energy storage structure $Q_0(t)$ compatible with abrupt changes from gating.

(Connection to Exponential Decay) The final statement just notes that if the initial V meets the conditions of Theorem 3.1 (strict passivity $\beta > 0$ and quadratic bounds), then Theorem 3.1 directly implies exponential decay for the unforced system, independent of the existence and properties of $V_{a,0}$.

Summary: This theorem provides a crucial bridge to the rigorous theory of passive LTV systems developed by Morandin & Hinsen [20]. It shows that the mere existence of any storage function V demonstrating passivity for the overall selective SSM guarantees that the unforced dynamics ($x = 0$) possess an underlying minimal energy storage function $V_{a,0}$ that is inherently quadratic, $V_{a,0}(t, h) = \frac{1}{2} h^H Q_0(t) h$. Critically, the defining matrix $Q_0(t)$ exhibits $\mathrm{AUC}_{\mathrm{loc}}$ regularity, a property robust enough to handle discontinuities introduced by the gating signal $\Delta(t)$. This establishes a foundational, well-behaved quadratic energy structure associated with the system's intrinsic dynamics, highlighting that even amidst complex input-driven parameter variations, the system's behavior from a given initial state $h(0)$ is governed by dynamics with this regular underlying structure when input is removed.

A.5 Proof of Theorem 4.1

(a) and (b): The premise is that $V_Q(t, h)$ satisfies the passivity inequality (12) for all admissible trajectories of the selective SSM (1). As argued in the proof of Theorem 3.3(a), any such trajectory includes the trajectories of the unforced LTV system (10) obtained by setting $x(t) \equiv 0$. Therefore, V_Q must also be a valid storage function for this passive LTV system (satisfying $V_Q(T, h(T)) - V_Q(t_0, h(t_0)) \leq 0$ along unforced trajectories).

The framework of Morandin & Hinsen [20] analyzes quadratic storage functions for passive LTV systems under the assumed L^p_{loc} regularity. Specifically, [20, Theorem 3.2] establishes necessary conditions for a quadratic form $V_Q(t, h) = \frac{1}{2} h^H Q(t) h$ to be a storage function. It implies that Q must be absolutely upper semi-continuous (which corresponds to $\mathrm{AUC}_{\mathrm{loc}}$ in their terminology). Furthermore, [20, Theorem 5.4(i)], when discussing the properties of the available storage (which provides bounds on any storage function), shows that the associated matrix function must have weakly monotonically non-increasing rank. Since V_Q is a storage function, it must be bounded by the maximal storage (related to available storage from the past) and bound the minimal available storage $V_{a,0}$ discussed in Theorem 3.3. The structural properties, particularly the non-increasing rank, apply to any quadratic storage function candidate within their framework. Therefore, $Q(t)$ must belong to $\mathrm{AUC}_{\mathrm{loc}}(\mathbb{T}, \mathbf{S}^n_+(\mathbb{C}))$ and its rank $r(t)$ must be weakly monotonically non-increasing.

Summary: This theorem reveals that the strong requirement of universal passivity guaranteed by a single V_Q immediately imposes significant structure on $Q(t)$ itself. It must possess $\mathrm{AUC}_{\mathrm{loc}}$ regularity, accommodating potential discontinuities but ensuring they behave in a controlled manner (e.g., jumps cannot increase energy in the Loewner sense). Crucially, the rank monotonicity implies that the dimension of the subspace captured by the energy function (its image) cannot increase over time.

This lays the groundwork for understanding irreversible effects, as the "energy-less" subspace (the kernel) can only grow or stay the same.

A.6 Proof of Theorem 4.2

(a) **Parametric LMI Condition:** The passivity inequality (12) holding for all trajectories is equivalent (under $\mathrm{AUC}_{\mathrm{loc}}$ regularity of Q) to its differential form holding almost everywhere along trajectories: $\frac{d}{dt}V_Q(t,h(t)) \le \mathrm{Re}\,\langle x(t), y(t)\rangle - \beta\|h(t)\|^2$. Substituting $V_Q = \frac{1}{2}h^H Q h$, $\dot{h} = A(t,x)h + B(t,x)x$, and $y = C(t,x)h$ leads to the quadratic inequality in h and x:

$$\frac{1}{2}h^H\dot{Q}h + \mathrm{Re}(h^H Q\dot{h}) \le \mathrm{Re}(x^H C(t,x)h) - \beta\|h\|^2 \tag{30}$$

$$\frac{1}{2}h^H\dot{Q}h + \mathrm{Re}(h^H Q(A(t,x)h + B(t,x)x)) \le \mathrm{Re}(x^H C(t,x)h) - \beta\|h\|^2 \tag{31}$$

Rearranging terms:

$$\frac{1}{2}h^H\dot{Q}h + \mathrm{Re}(h^H QAh) + \mathrm{Re}(h^H QBx) - \mathrm{Re}((C^H x)^H h) + \beta h^H h \le 0 \tag{32}$$

$$\frac{1}{2}h^H(\dot{Q} + QA + A^H Q + 2\beta I)h + \mathrm{Re}(h^H(QB - C^H)x) \le 0 \tag{33}$$

This inequality must hold for all $h \in \mathbb{C}^N$ and all admissible $x \in \mathbb{C}^{d_{\mathrm{in}}}$ at almost every time t. This is precisely the condition encoded by the negative semidefiniteness of the LMI matrix $\mathcal{L}(t,x)$ defined in (13). Since V_Q must work for any input trajectory, the LMI must hold parametrically for all input values x that can occur at time t. This leads directly to the LMI formulation (setting $D = 0$):

$$\begin{bmatrix} \dot{Q}(t) + Q(t)A(t,x) + A(t,x)^H Q(t) + 2\beta I & Q(t)B(t,x) - C(t,x)^H \\ B(t,x)^H Q(t) - C(t,x) & 0 \end{bmatrix} \preceq 0 \tag{34}$$

(b) **Universal Kernel Condition for C:** Let $v \in \mathrm{Ker}(Q(t))$ at a time t where the LMI (13) holds. Consider the augmented vector $z = \begin{bmatrix} v \\ w \end{bmatrix}$ for any $w \in \mathbb{C}^{d_{\mathrm{in}}}$. The LMI implies $z^H\mathcal{L}(t,x)z \le 0$.

$$\begin{aligned} z^H\mathcal{L}(t,x)z &= \begin{bmatrix} v^H & w^H \end{bmatrix} \begin{bmatrix} \dot{Q} + QA + A^H Q + 2\beta I & QB - C^H \\ B^H Q - C & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \\ &= v^H(\dot{Q} + QA + A^H Q + 2\beta I)v + v^H(QB - C^H)w + w^H(B^H Q - C)v + w^H(0)w \end{aligned}$$

Since $v \in \mathrm{Ker}(Q(t))$, we have $Q(t)v = 0$ and $v^H Q(t) = 0$. The expression simplifies to:

$$\begin{aligned} &= v^H(\dot{Q} + 0 + 0 + 2\beta I)v + v^H(0 - C^H)w + w^H(0 - C)v \\ &= v^H\dot{Q}v + 2\beta\|v\|^2 - v^H C^H w - w^H C v \\ &= v^H\dot{Q}v + 2\beta\|v\|^2 - 2\,\mathrm{Re}(w^H C v) \end{aligned}$$

This quadratic form in w must be ≤ 0. Considering the 2×2 projection onto v and w space: $\begin{bmatrix} v^H(\dot{Q} + QA + A^H Q + 2\beta I)v & v^H(QB - C^H)w \\ w^H(B^H Q - C)v & 0 \end{bmatrix} = \begin{bmatrix} v^H\dot{Q}v + 2\beta\|v\|^2 & -v^H C(t,x)^H w \\ -w^H C(t,x)v & 0 \end{bmatrix} \preceq 0$. For this 2×2 matrix to be negative semidefinite, the diagonal elements must be non-positive ($v^H\dot{Q}v + 2\beta\|v\|^2 \le 0$, which holds as seen in Theorem 4.3, and $0 \le 0$), and the determinant must be non-negative. The determinant is $(v^H\dot{Q}v + 2\beta\|v\|^2)(0) - |-w^H C(t,x)v|^2 = -|w^H C(t,x)v|^2$. So we need $-|w^H C(t,x)v|^2 \ge 0$. This can only hold if $|w^H C(t,x)v|^2 = 0$ for all $w \in \mathbb{C}^{d_{\mathrm{in}}}$. This implies $C(t,x)v = 0$. Since this must hold for the $C(t,x)$ corresponding to any admissible x at time t, we conclude $\forall v \in \mathrm{Ker}(Q(t))$, $C(\Delta(t),x)v = 0$ for all admissible x, a.e. t.

(c) **Implicit Constraints on A and B:** The validity of the LMI (13) for all x directly imposes constraints on $A(t,x)$ via the $(1,1)$ block and couples $B(t,x)$ to $C(t,x)$ (which is already constrained by part (b)) via the off-diagonal blocks and $Q(t)$. These ensure that the dynamics generated by any input x remain compatible with the energy storage/dissipation defined by V_Q.

Summary: This theorem translates the abstract requirement of universal passivity into a concrete parametric LMI (13). This LMI must hold not just for a specific input or for the unforced system,

but simultaneously for all possible input values x that the gating mechanism might encounter at any given time t. A striking consequence is the universal kernel condition (14): any state direction v that is considered "energy-less" by the storage function (i.e., $v \in \mathrm{Ker}(Q(t))$) must be rendered unobservable at the output ($C(t,x)v = 0$), irrespective of the specific input x driving the gating. This imposes a strong limitation on the gating mechanism: it cannot arbitrarily change the output matrix C in response to input x in a way that would make these energy-less states visible. Passivity demands consistency between the energy accounting (Q) and observability (C).

A.7 Proof of Theorem 4.3

1. **Kernel Non-Shrinking:** This follows directly from Theorem 4.1(b), which established that $\mathrm{rank}(Q(t))$ is weakly monotonically non-increasing. Since $\dim(\mathrm{Ker}(Q(t))) = N - \mathrm{rank}(Q(t))$, a non-increasing rank implies a non-decreasing kernel dimension. The inclusion $\mathrm{Ker}(Q(t_1)) \subseteq \mathrm{Ker}(Q(t_2))$ for $t_1 \leq t_2$ is a consequence of the properties of $\mathrm{AUC}_{\mathrm{loc}}$ functions shown in [20, Section 4].

2. **Energy Consistency within the Kernel:** This condition was derived within the proof of Theorem 4.2(b) by evaluating the $(1,1)$ block of the parametric LMI (13) for a vector $h(t) \in \mathrm{Ker}(Q(t))$. The $(1,1)$ block inequality is $\dot{Q}(t)+Q(t)A(t,x)+A(t,x)^H Q(t)+2\beta I \preceq 0$. Pre- and post-multiplying by $h(t)^H$ and $h(t)$ respectively, and using $Q(t)h(t) = 0$, yields $h(t)^H \dot{Q}(t) h(t) + 2\beta \|h(t)\|^2 \leq 0$. This inequality must hold independently of x because it follows from the LMI which holds parametrically.

3. **Constraint on Dynamics Violating Kernel Structure:** If dynamics induced by some $x^*(t)$ were fundamentally incompatible with the passivity condition guaranteed by V_Q (e.g., by attempting to move a state out of the kernel in an "energy-creating" way relative to V_Q), it would necessarily cause the parametric LMI (13) to fail for $x = x^*(t)$. This contradicts the core assumption that V_Q ensures passivity universally. Therefore, all admissible dynamics under the universal V_Q assumption must inherently respect the energy balance constraints, including those within the kernel.

Summary: This theorem formalizes the notion of "irreversible forgetting" within the framework of a universal quadratic storage function. The non-shrinking kernel (Property 1) implies that once a state direction is deemed irrelevant or forgotten from an energy perspective (i.e., enters $\mathrm{Ker}(Q(t))$), it structurally remains so according to the fixed energy measure $Q(t)$. The gating mechanism cannot manipulate $Q(t)$ to "un-forget" this mode. Furthermore, Property 2 ensures that the system dynamics and the evolution of $Q(t)$ itself must maintain energy consistency within this forgotten subspace, respecting the required dissipation rate β. Any gating strategy $x(t)$ must induce dynamics $(A(t,x), B(t,x), C(t,x))$ compatible with these constraints (Property 3). This provides a lens for analyzing robust memory properties: modes in $\mathrm{Ker}(Q(t))$ are stably forgotten. It also offers potential insights into phenomena like catastrophic forgetting in learning systems; if learning adapts a $Q(t)$-like structure, changes that violate kernel inertness or energy consistency might be necessary to learn new conflicting information, potentially disrupting the established passive structure.

A.8 Proof of Theorem 5.1

We analyze the time derivative of the Lyapunov function candidate $V_Q(t, h(t)) = \frac{1}{2} h(t)^H Q(t) h(t)$ along the trajectories of the full system (1). Since $Q \in W^{1,1}_{\mathrm{loc}}$, its derivative $\dot{Q}(t)$ exists a.e. and is in

L^1_{loc}. Using the chain rule and substituting $\dot{h}(t) = A(t, x(t))h(t) + B(t, x(t))x(t)$:

$$\begin{aligned}
\frac{d}{dt}V_Q(t, h(t)) &= \frac{1}{2}h^H\dot{Q}h + \frac{1}{2}\dot{h}^H Qh + \frac{1}{2}h^H Q\dot{h} \\
&= \frac{1}{2}h^H\dot{Q}h + \operatorname{Re}\left(h^H Q\dot{h}\right) \\
&= \frac{1}{2}h^H\dot{Q}h + \operatorname{Re}\left(h^H Q\left(A(t, x(t))h + B(t, x(t))x(t)\right)\right) \\
&= \underbrace{\frac{1}{2}h^H\left(\dot{Q}(t) + Q(t)A(t, x(t)) + A(t, x(t))^H Q(t)\right)h}_{\text{Homogeneous Part}} \\
&\quad + \underbrace{\operatorname{Re}\left(h^H Q(t)B(t, x(t))x(t)\right)}_{\text{Input Part}}
\end{aligned}$$

Here, $A(t, x(t))$ stands for $A(\Delta(t), x(t))$, and similarly for B.

Now, we use the assumptions: 1. Homogeneous Part: By the uniform dissipativity assumption (16), which holds for the specific value $x = x(t)$ occurring at time t, we have:

$$\frac{1}{2}h^H\left(\dot{Q}(t) + Q(t)A(t, x(t)) + A(t, x(t))^H Q(t)\right)h \leq \frac{1}{2}h^H(-2\delta Q(t))h = -\delta h^H Q(t)h = -2\delta V_Q(t, h(t))$$

2. Input Part: We bound this term using Cauchy-Schwarz and the uniform bounds on Q and B:

$$\begin{aligned}
|\operatorname{Re}(h^H Q(t)B(t, x(t))x(t))| &\leq \|h^H Q(t)B(t, x(t))x(t)\| \\
&\leq \|h(t)\|\|Q(t)\|_2\|B(t, x(t))\|_2\|x(t)\| \\
&\leq \|h(t)\|k_2 M_B\|x(t)\| \quad \text{(using } \|Q(t)\|_2 \leq k_2 \text{ since } Q \preceq k_2 I\text{)}
\end{aligned}$$

We relate $\|h(t)\|$ back to $V_Q(t, h(t))$ using the lower bound $Q(t) \succeq k_1 I$: $V_Q(t, h(t)) = \frac{1}{2}h^H Qh \geq \frac{1}{2}h^H(k_1 I)h = \frac{k_1}{2}\|h\|^2$. Thus, $\|h(t)\| \leq \sqrt{\frac{2V_Q(t,h(t))}{k_1}}$. Substituting this into the bound for the input part:

$$|\operatorname{Re}(h^H QBx)| \leq \sqrt{\frac{2V_Q}{k_1}}k_2 M_B\|x(t)\| \tag{35}$$

Combining the bounds for the homogeneous and input parts:

$$\frac{d}{dt}V_Q(t, h(t)) \leq -2\delta V_Q(t, h(t)) + \left(\sqrt{\frac{2}{k_1}}k_2 M_B\right)\|x(t)\|\sqrt{V_Q(t, h(t))} \tag{36}$$

Let $\Phi(t) = V_Q(t, h(t)) \geq 0$. Let $K = \sqrt{\frac{2}{k_1}}k_2 M_B \geq 0$. The inequality is:

$$\dot{\Phi}(t) \leq -2\delta\Phi(t) + K\|x(t)\|\sqrt{\Phi(t)} \tag{37}$$

This is a standard differential inequality form used in ISS proofs [25, 16]. Consider $\Psi(t) = \sqrt{\Phi(t)}$. For $\Phi > 0$, $\dot{\Psi} = \frac{1}{2\sqrt{\Phi}}\dot{\Phi}$.

$$\dot{\Psi} \leq \frac{1}{2\sqrt{\Phi}}(-2\delta\Phi + K\|x\|\sqrt{\Phi}) = -\delta\sqrt{\Phi} + \frac{K}{2}\|x\| = -\delta\Psi + \frac{K}{2}\|x\| \tag{38}$$

So, $\dot{\Psi}(t) \leq -\delta\Psi(t) + \frac{K}{2}\|x(t)\|$. By the Comparison Principle (integrating factor method or standard lemma, check derivation in Appendix A.9):

$$\Psi(t) \leq e^{-\delta(t-t_0)}\Psi(t_0) + \int_{t_0}^{t} e^{-\delta(t-\tau)}\frac{K}{2}\|x(\tau)\|d\tau \tag{39}$$

Let $\|x\|_{[t_0,t],\infty} = \sup_{t_0 \le \tau \le t} \|x(\tau)\|$.

$$\begin{aligned}\int_{t_0}^{t} e^{-\delta(t-\tau)} \frac{K}{2} \|x(\tau)\| d\tau &\le \frac{K}{2} \|x\|_{[t_0,t],\infty} \int_{t_0}^{t} e^{-\delta(t-\tau)} d\tau \\ &= \frac{K}{2} \|x\|_{[t_0,t],\infty} \left[\frac{1}{\delta} e^{-\delta(t-\tau)} \right]_{\tau=t_0}^{\tau=t} \\ &= \frac{K}{2\delta} \|x\|_{[t_0,t],\infty} (1 - e^{-\delta(t-t_0)}) \\ &\le \frac{K}{2\delta} \|x\|_{[t_0,t],\infty}\end{aligned}$$

Therefore,

$$\Psi(t) \le e^{-\delta(t-t_0)} \Psi(t_0) + \frac{K}{2\delta} \|x\|_{[t_0,t],\infty} \tag{40}$$

Substituting back $\Psi = \sqrt{V_Q}$:

$$\sqrt{V_Q(t, h(t))} \le e^{-\delta(t-t_0)} \sqrt{V_Q(t_0, h(t_0))} + \frac{K}{2\delta} \|x\|_{[t_0,t],\infty} \tag{41}$$

Using the quadratic bounds $k_1 I \preceq Q(t) \preceq k_2 I$: $\sqrt{V_Q} \ge \sqrt{k_1/2}\|h\|$ and $\sqrt{V_Q} \le \sqrt{k_2/2}\|h\|$.

$$\begin{aligned}\sqrt{k_1/2}\|h(t)\| &\le \sqrt{V_Q(t, h(t))} \\ &\le e^{-\delta(t-t_0)} \sqrt{V_Q(t_0, h(t_0))} + \frac{K}{2\delta} \|x\|_{[t_0,t],\infty} \\ &\le e^{-\delta(t-t_0)} \sqrt{k_2/2} \|h(t_0)\| + \frac{K}{2\delta} \|x\|_{[t_0,t],\infty}\end{aligned}$$

Dividing by $\sqrt{k_1/2}$:

$$\|h(t)\| \le \sqrt{\frac{k_2}{k_1}} e^{-\delta(t-t_0)} \|h(t_0)\| + \frac{K}{2\delta\sqrt{k_1/2}} \|x\|_{[t_0,t],\infty} \tag{42}$$

This is the ISS estimate (17) with:

- $\tilde{C} = \sqrt{k_2/k_1} \ge 1$
- $\tilde{\gamma} = \delta > 0$
- $\sigma(s) = K's$, where $K' = \frac{K}{2\delta\sqrt{k_1/2}} = \frac{(\sqrt{2/k_1} k_2 M_B)}{2\delta\sqrt{k_1/2}} = \frac{k_2 M_B}{\delta k_1} \ge 0$.

Since $\sigma(s) = K's$ is a class $\mathcal{K}$ function (specifically, linear), the system is ISS.

Summary: This theorem provides a powerful sufficient condition for ensuring the robust stability of the selective SSM in the face of arbitrary bounded inputs. The core requirement is the existence of a single quadratic Lyapunov function V_Q (which must be uniformly bounded and positive definite) whose time derivative decreases at a guaranteed rate ($-\delta V_Q$), regardless of which specific dynamics $A(t, x)$ are activated by the input $x(t)$ via the gating mechanism. This condition (16) essentially demands that every possible "operating mode" induced by the input is individually exponentially stable, and that this stability is certified by the same Lyapunov function V_Q with a uniform decay rate δ. If this strong condition holds, and the input matrix $B(t, x)$ is bounded, the theorem guarantees ISS. This elegantly connects the uniform stability properties of all possible input-selected local dynamics to the global robustness of the overall selective system against external disturbances or inputs. Finding such a common V_Q and verifying the uniform dissipativity condition provides a direct pathway to certifying the robust stability of complex selective SSM architectures.

A.9 Explanation of Comparison Principle and derivation of Eq. (39)

Let's start with the differential inequality we derived for $\Psi(t)$:

$$\dot{\Psi}(t) \le -\delta \Psi(t) + \frac{K}{2} \|x(t)\| \tag{43}$$

This can be rewritten as:

$$\dot{\Psi}(t) + \delta\Psi(t) \leq \frac{K}{2}\|x(t)\| \tag{44}$$

where $\delta > 0$. The inequality for $\Psi(t)$ is a standard first-order linear differential inequality. Its solution bound can be obtained either by the explicit method of integrating factors or by invoking a suitable Comparison Lemma (which is itself often proved using integrating factors or similar techniques). Both approaches lead to the same expression for the upper bound of $\Psi(t)$.

- Method 1: Using an Integrating Factor (Standard ODE Technique)

This is a common method for solving first-order linear ordinary differential equations. The idea is to multiply the equation by a factor that makes the left-hand side the derivative of a product. The integrating factor for an equation of the form $\dot{y} + p(t)y = q(t)$ is $e^{\int p(t)dt}$. In our case, $p(t) = \delta$ (a constant). So the integrating factor is $e^{\int \delta dt} = e^{\delta t}$.

Multiply both sides of (44) by $e^{\delta t}$:

$$e^{\delta t}\dot{\Psi}(t) + \delta e^{\delta t}\Psi(t) \leq e^{\delta t}\frac{K}{2}\|x(t)\| \tag{45}$$

Recognize that the left-hand side is the derivative of a product:

$$\frac{d}{dt}\left(e^{\delta t}\Psi(t)\right) = e^{\delta t}\dot{\Psi}(t) + \left(\frac{d}{dt}e^{\delta t}\right)\Psi(t) = e^{\delta t}\dot{\Psi}(t) + \delta e^{\delta t}\Psi(t) \tag{46}$$

So, the inequality becomes:

$$\frac{d}{dt}\left(e^{\delta t}\Psi(t)\right) \leq e^{\delta t}\frac{K}{2}\|x(t)\| \tag{47}$$

Now, integrate both sides from an initial time t_0 to a general time t (let's use τ as the integration variable to avoid confusion):

$$\int_{t_0}^{t} \frac{d}{d\tau}\left(e^{\delta\tau}\Psi(\tau)\right)d\tau \leq \int_{t_0}^{t} e^{\delta\tau}\frac{K}{2}\|x(\tau)\|d\tau \tag{48}$$

By the Fundamental Theorem of Calculus, the left side is:

$$\left[e^{\delta\tau}\Psi(\tau)\right]_{t_0}^{t} = e^{\delta t}\Psi(t) - e^{\delta t_0}\Psi(t_0) \tag{49}$$

So,

$$e^{\delta t}\Psi(t) - e^{\delta t_0}\Psi(t_0) \leq \int_{t_0}^{t} e^{\delta\tau}\frac{K}{2}\|x(\tau)\|d\tau \tag{50}$$

Now, solve for $\Psi(t)$:

$$e^{\delta t}\Psi(t) \leq e^{\delta t_0}\Psi(t_0) + \int_{t_0}^{t} e^{\delta\tau}\frac{K}{2}\|x(\tau)\|d\tau \tag{51}$$

Multiply by $e^{-\delta t}$ (which is positive):

$$\Psi(t) \leq e^{-\delta t}e^{\delta t_0}\Psi(t_0) + e^{-\delta t}\int_{t_0}^{t} e^{\delta\tau}\frac{K}{2}\|x(\tau)\|d\tau \tag{52}$$

$$\Psi(t) \leq e^{-\delta(t-t_0)}\Psi(t_0) + \int_{t_0}^{t} e^{-\delta(t-\tau)}\frac{K}{2}\|x(\tau)\|d\tau \tag{53}$$

This gives us the result we want.

- Method 2: Direct Application of a Comparison Lemma (e.g., a form of Grönwall's Lemma)

Many control theory and differential equations textbooks state a "Comparison Lemma" or a specific form of Grönwall's Lemma that directly applies. For example, a common version states: If $\dot{u}(t) \leq a(t)u(t) + b(t)$ and $\dot{v}(t) = a(t)v(t) + b(t)$ with $u(t_0) \leq v(t_0)$, then $u(t) \leq v(t)$ for $t \geq t_0$. If we have an inequality $\dot{u}(t) \leq au(t) + b(t)$ (where a is constant), the solution to the corresponding equality $\dot{v}(t) = av(t) + b(t)$ with $v(t_0) = u(t_0)$ is given by the variation of parameters formula: $v(t) = e^{a(t-t_0)}v(t_0) + \int_{t_0}^{t} e^{a(t-\tau)}b(\tau)d\tau$. Then, by the Comparison Principle, $u(t) \leq v(t)$.

In our case, from (43): $\dot{\Psi}(t) \leq -\delta\Psi(t) + \underbrace{\frac{K}{2}\|x(t)\|}_{b(t)}$ Here, the coefficient of $\Psi(t)$ is $a = -\delta$. The "forcing term" is $b(t) = \frac{K}{2}\|x(t)\|$.

So, applying the solution form for $v(t)$ directly with $a = -\delta$:

$$\Psi(t) \leq e^{-\delta(t-t_0)}\Psi(t_0) + \int_{t_0}^{t} e^{-\delta(t-\tau)} \left(\frac{K}{2}\|x(\tau)\|\right) d\tau \tag{54}$$

This again yields the desired result.